A NOTE ON CONVERGENCE OF SUB-MARTINGALES[1]

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A number of conditions for convergence of sub-martingales are known today. The earliest of these in Doob's theorem [2] which requires an upper bound on the expectations of x_n. Snell [4] then generalized Doob's result by conditioning the expectations. Chow [1] proved a different generalization based on random times. It was suggested by Loève that the two results could be combined to yield a further generalization. The theorem presented here is of this nature. The proof adheres very closely to Chow's approach.

Lemma. *Let* $(x_n, n \geqq 1, \mathcal{A}_n)$ *be a sub-martingale sequence. Then for all* k, n, m, $k \leqq n \leqq m$:

$$E(I_{(C_{kn}-D_{nm})}x_m \mid \mathcal{A}_k) \geqq P(C_{kn} \mid \mathcal{A}_k), \text{ a.s.}$$

where $C_{kn} = \{x_i \geqq 1$ *for some* i *such that* $k \leqq i < n\}$ *and* $D_{kn} = \{x_i \leqq 0$ *for some* i *such that* $k \leqq i < n\}$.

Proof. Define t_1 and t_2 by

$$\begin{aligned} t_1 &= \inf\{i : x_i \geqq 1\} && \text{on } C_{kn} \\ &= n && \text{on } C_{kn}^c, \\ t_2 &= \inf\{i : x_i \leqq 0\} && \text{on } D_{nm}C_{kn} \\ &= n && \text{on } C_{kn}^c \\ &= m && \text{on } C_{kn} - D_{nm}C_{kn}. \end{aligned}$$

Then t_1 and t_2 are $(\mathcal{A}_n)$-times and $m \geqq t_2 \geqq n \geqq t \geqq k$, hence $E(x_{t_2} \mid \mathcal{A}_{t_1}) \geqq E(x_{t_1} \mid \mathcal{A}_{t_1})$ or $E(x_{t_2} \mid \mathcal{A}_k) = E(E(x_{t_2} \mid \mathcal{A}_{t_1}) \mid \mathcal{A}_k) \geqq E(x_{t_1} \mid \mathcal{A}_k)$, since $\mathcal{A}_k \subset \mathcal{A}_{t_1}$. Also $I_{D_{nm}}x_{t_2} \leqq 0$, therefore

$$\begin{aligned} E(I_{(C_{kn}-D_{nm})C_{kn}}x_m \mid \mathcal{A}_k) &= E(I_{D_{kn}}x_{t_2} \mid \mathcal{A}_k) - E(I_{D_{nm}C_{kn}}x_{t_2} \mid \mathcal{A}_k) \\ &\geqq E(I_{C_{kn}}x_{t_1} \mid \mathcal{A}_k) \geqq E(I_{C_{kn}} \mid \mathcal{A}_k) = P(C_{kn} \mid \mathcal{A}_k). \end{aligned}$$

Definition. A random variable is called an $(\mathcal{A}_n)$-time (martingale time, stopping time) iff $\{t = n\} \varepsilon \mathcal{A}_n$ (Loève [3], p. 530).

Theorem. *Let* $[x_n, \mathcal{A}_n, n \geqq 1)$ *be a sub-martingale sequence. Let* F *be a measurable set such that for each* $(\mathcal{A}_n)$*-time* t *there exists* $\{n_j\}_{j=1}^{\infty}$ *such that*

$$\lim_{n_j \to \infty} P(\{E(I_{[n \leqq t < \infty]}x_t^+ \mid \mathcal{A}_{n_j}) = \infty\}F) = 0.$$

Then $\lim x_n$ *exists* a.e. *on* F.

Proof. It will be shown that if $\lim x_n$ does not exist there exists an $(\mathcal{A}_n)$-time

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