EXTREMAL PROCESSES[1]

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1. Introduction. Suppose $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables, and $M_n = \max(X_1, \cdots, X_n)$, $n = 1, 2, \cdots$. Necessary and sufficient conditions for the existence of sequences of constants $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n\to\infty} P\{(M_n - a_n)/b_n < t\} = F(t)$ should exist for every real t, with F being a non-degenerate c.d.f., are well known. It is also known that up to a location or scale parameter, F must have one of the following three forms:

$$
\begin{aligned}
F_1(t, \lambda) &= 0, && t \leqq 0,\\
&= \exp -\lambda t^{-\alpha}, && t > 0;\\
F_2(t, \lambda) &= \exp -\lambda(-t)^{\alpha}, && t \leqq 0,\\
&= 1, && t > 0;\\
F_3(t, \lambda) &= \exp[-\lambda(\exp -t)], && -\infty < t < \infty;
\end{aligned} \tag{1.1}
$$

λ being a positive constant, [3]. (In deference to a long-established usage, we wish to point out that the usual notations for F_1, F_2, F_3 are Φ_α, Ψ_α, and Λ respectively.)

The purpose of this paper is to study the stochastic processes, $Y(t)$, which are in a natural sense the "limits," as $n \to \infty$, of the processes

$$Y_n(t) = (M_{[tn]+1} - a_n)/b_n, \qquad 0 \leqq t < \infty.$$

(A study of the passage to the limit is given in the following paper by John Lamperti, [4].) The "limiting" process, $Y(t)$, will be defined rigorously in Section 3. There are, of course, three possible processes, according to whether the X_i's belong to the "laws of attraction" F_1, F_2, or F_3 respectively. We will refer to these processes as *extremal processes* of types 1, 2, 3.

Extremal processes bear a natural analogy with stable processes. Thus, suppose that $U_1, U_2, \cdots$ is a sequence of independent and identically distributed random variables, $S_n = U_1 + \cdots + U_n$, $n = 1, 2, \cdots$, and $\{c_n\}$ and $\{d_n\}$ are sequences of constants such that $\{(S_n - c_n)/d_n\}$ converges in law to a non-degenerate distribution, which is necessarily a distribution of stable type. Then if we define a process

$$Z_n(t) = (S_{[tn]} - c_n)/d_n, \qquad 0 \leqq t < \infty,$$

Received September 17, 1963; revised 25 May 1964.

[1] Research sponsored by Air Force Office of Scientific Research Grant No. AF-AFOSR-67-63.