A TRUSTWORTHY JACKKNIFE[1]

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1. Introduction. A method for the reduction of bias in parametric estimation was introduced by Quenouille [6], and its properties were studied in some specific situations by Quenouille [7] and Durbin [4]. Tukey [9] proposed the general use of this technique in order to (a) reduce the bias and (b) obtain approximate confidence intervals in problems where standard statistical procedures may not exist or are difficult to apply. Tukey adopted the name of "jackknife" for this procedure, since a boy scout's jackknife is symbolic of a rough-and-ready instrument capable of being utilized in all contingencies and emergencies.

The jackknife procedure depends upon judiciously dividing the data into groups, obtaining estimates from combinations of these groups, and then averaging the estimates. Let θ be the unknown parameter, and let $(X_1, \cdots, X_N)$ be a sample of N independent, identically distributed observations with cdf F_θ, which depends upon θ. Suppose a reasonably good (but biased) method of estimating θ is available. Further suppose the data can be divided into n groups of size k $(N = nk)$, i.e., $(X_1, \cdots, X_k; X_{k+1}, \cdots, X_{2k}; \cdots; X_{(n-1)k+1}, \cdots, X_{nk})$. Denote by $\hat{\theta}_{n-1}^i$, $i = 1, \cdots, n$, the estimate of θ obtained by deleting the ith group and estimating θ from the other $(n-1)k$ observations, e.g., $\hat{\theta}_{n-1}^1 = \hat{\theta}(X_{k+1}, \cdots, X_{kn})$. Let $\hat{\theta}_n^0$ be the estimate of θ based on all nk observations. Form the new estimates (called "pseudo-values" by Tukey)

$$\hat{\theta}_i = n\hat{\theta}_n^0 - (n-1)\hat{\theta}_{n-1}^i, \qquad i = 1, \cdots, n. \tag{1}$$

The jackknife estimate of θ is the average of the $\hat{\theta}_i$, $i = 1, \cdots, n$, i.e.,

$$\hat{\theta} = n^{-1}\sum_1^n \hat{\theta}_i = n\hat{\theta}_n^0 - (n-1)\hat{\theta}_{n-1}^{\cdot}, \tag{2}$$

where $\hat{\theta}_{n-1}^{\cdot} = (\sum_1^n \hat{\theta}_{n-1}^i)/n$.

The jackknife $\hat{\theta}$ exactly eliminates the $1/n$ term from any bias. For if

$$E(\hat{\theta}_n^0) = \theta + a/kn + b/(kn)^2 + \cdots, \tag{3}$$

for all n and k, then

$$\begin{aligned} E(\hat{\theta}) = n(\theta + a/kn + b/(kn)^2 + \cdots) - (n-1)(\theta + a/k(n-1) \\ + b/(k(n-1))^2 + \cdots) = \theta - b/k^2n(n-1) + \cdots. \end{aligned} \tag{4}$$

Quenouille [7] and Durbin [4] have shown that in certain ratio problems this

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