JOINT DISTRIBUTIONS WITH PRESCRIBED MOMENTS[1]

By Howard Rumsey, Jr. and Edward C. Posner

Jet Propulsion Laboratory, California Institute of Technology

1. Introduction. Suppose two random variables X and Y have a joint distribution, whereas only the marginal distributions are known, and also certain other information. (For example, let the two random variables represent height from reference level at two randomly chosen points a given distance apart on the Moon's surface. The distribution of heights at a single point is fairly well-known, and radar data also determine the average value of the absolute value of the difference in heights, that is, $E(|X - Y|)$, [5].) It is then desired to make inferences about the joint distribution of X and Y. This article adopts the principle of maximum entropy [2] to select one plausible joint distribution from the various possible ones consistent with the given marginal distributions and a given set of cross moments. It is shown that a joint distribution of maximum entropy exists and is unique, under very general conditions. The unique joint distribution is shown to have a special form, by means of which it is shown that the unique joint distribution can be obtained as the solution of a certain set of non-linear integral equations. The techniques involve functional analysis and the calculus of variations, as well as standard properties of the entropy functional.

2. Problem statement. We shall solve problems of the following kind. We are given two random variables X and Y and information about their joint density function $f = f(x, y)$. In particular, we are given the marginal density functions of X and Y, say $p(x)$ and $q(y)$. Thus,

$$\int_{-\infty}^{\infty} f(x, y)\, dy = p(x), \qquad \int_{-\infty}^{\infty} f(x, y)\, dx = q(y),$$

where both equations hold almost everywhere (a.e.). In addition, we are given a finite number (possibly zero) of other linear constraints which the density function f must satisfy. These constraints are to be expressed in the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) r_j(x, y)\, dx\, dy = \rho_j\,, \qquad j = 1, \cdots, k,$$

where the r_j are given functions and the ρ_j are given constants. The problem is to find that joint density function f which satisfies all these constraint equations and which in addition has maximum entropy

$$H(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \log f^{-1}\, dx\, dy.$$

Received 7 August 1964.

[1] This paper represents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.