MAXIMUM LIKELIHOOD ESTIMATION FOR DISTRIBUTIONS WITH MONOTONE FAILURE RATE

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1. Introduction. Given a set of observations $X_1, \cdots, X_n$ from a common distribution function F, it is natural in the absence of additional information to estimate F by the usual empirical distribution function. However, one would not use this estimator if there were at hand sufficient *a priori* information about the distribution F, e.g., that F is a member of a given parameteric class such as the normal. In this paper, we examine an intermediate case, the case that F is known to have monotone failure rate. Using the idea of maximum likelihood, Grenander [10] derives an estimator for F in the case of increasing failure rate which itself has increasing failure rate. We discuss this case and also obtain estimators in the case of decreasing failure rate. We show that these estimators are consistent.

2. Properties of IFR distributions, and formulation of the problem. The failure rate (or hazard rate) r of a distribution F having derivative f is defined by $r(x) = f(x)/[1 - F(x)]$, for $F(x) < 1$. It is easy to verify that if r is increasing, then $R(x) \equiv -\log[1 - F(x)]$ is convex on the support of F, an interval. (Throughout this paper we write "increasing" for "nondecreasing" and "decreasing" for "nonincreasing.") Whether f exists or not, we say that F has increasing failure rate (IFR) if the support of F is of the form $[\alpha, \beta]$, $-\infty \leqq \alpha \leqq \beta \leqq \infty$, and if R is convex on $[\alpha, \beta)$. The importance of the IFR property and its applications to life testing and reliability are discussed in [3], [4].

If F is IFR and $F(z) < 1$, then F is absolutely continuous on $(-\infty, z)$. To see this, choose $\epsilon > 0$ and points $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m \leqq z$ satisfying $\sum_1^m (\beta_i - \alpha_i) < \epsilon/r^+(z)$, where $r^+(z) = \lim_{\delta \downarrow 0} [R(z + \delta) - R(z)]/\delta$ exists finitely since R is convex. Then

$$\sum_1^m |R(\beta_i) - R(\alpha_i)| = \sum_1^m \frac{R(\beta_i) - R(\alpha_i)}{\beta_i - \alpha_i} (\beta_i - \alpha_i) \leqq r^+(z) \sum_1^m (\beta_i - \alpha_i) \leqq \epsilon.$$

Thus R is absolutely continuous on $(-\infty, z)$, and the result follows. Note, however, that F may have a jump at the right-hand endpoint of its interval of support.

For convenience, if F is IFR we define $r(x) = \infty$ for all x such that $F(x) = 1$. Note that for any distribution F and any x for which r is finite on $(-\infty, x)$, we have

$$(2.1) \qquad 1 - F(x) = \exp[-R(x)] = \exp[-\textstyle\int_{-\infty}^x r(z)\, dz].$$

Further properties of IFR distributions have been discussed in [2].

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