INTEGRAL KERNELS AND INVARIANT MEASURES FOR MARKOFF TRANSITION FUNCTIONS[1]

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1. Introduction. An important question concerning Markoff transition functions is, when do they possess invariant measures? One aspect of this question is the following: given a measure μ, when will P possess a nontrivial invariant measure $\epsilon \succ \mu$? If infinite ϵ is permitted, then the question becomes a more difficult one.

Harris [4] showed that if μ is a separable measure such that for each set A with $\mu(A) > 0$ and every x, the probability of ultimately getting from x to A is one, then there is a unique σ-finite invariant measure ϵ, and $\epsilon \succ \mu$. In [2], the present author attempted to replace this by some sort of almost-everywhere type of assumption (μ-recurrence). The key point seemed to be to require that $\sum_{n=0}^{\infty} 2^{-n}P^n$ consist partly of an integral operator (an assumption which was an automatic consequence of Harris's hypothesis). A theorem was proven there for the more general case of μ-conservative processes, but the assumptions were stronger than necessary. Recently, R. Isaac [7] proved the existence of an invariant measure in the μ-recurrent case, making much weaker assumptions about the integral operator part. He was unable, however, to show the relation between μ and the invariant measure.

In the present paper, we show under Isaac's hypothesis that his invariant measure is equivalent to $\sum_{n=0}^{\infty} 2^{-n}\mu P$ (Theorem 4). Actually, a theorem is proven for the more general μ-conservative case (Corollary to Theorem 4), but this turns out to be easy, for the following rather surprising reason. While in general a μ-conservative transition operator is some sort of integral average of recurrent operators, the presence of a nontrivial integral operator part forces this integral average to be a *discrete* direct sum (Corollary to Theorem 1). In the process of showing Theorem 4, it proves convenient to find out more precisely what the integral operator part of $\sum 2^{-n}P^n$ is like. This is done in Theorem 2.

2. The μ-nonsingular part of P. Let $\mathfrak{X}$ be a σ-algebra on a set X. Let P be a *subtransition* function, i.e. a function on $X \times \mathfrak{X}$ which is, for each $x \in X$, a nonnegative measure on $\mathfrak{X}$ of total mass $\leqq 1$, and for each $A \in \mathfrak{X}$, an $\mathfrak{X}$-measurable function. P induces an operator on $\mathfrak{L}_\infty(\mathfrak{X})$, by the rule $Pf(x) = \int f(y)P(x, dy)$, and also an operator on the nonnegative measures on $\mathfrak{X}$, by the rule

$$\mu P(A) = \int P(x, A)\mu(dy).$$

Let μ be a fixed σ-finite measure on $\mathfrak{X}$. Then $P(x, \cdot)$ has a unique decomposition

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