# ON THE ASYMPTOTIC BEHAVIOR OF BAYES ESTIMATES
# IN THE DISCRETE CASE II[1]

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**1. Introduction.** This article proves: when sampling from a countably infinite population with unknown distribution, for all but a set of priors of the first category, Bayes estimates are consistent only at a set of distributions of the first category. That is, for essentially all priors the Bayes estimates are consistent essentially nowhere. This article is independent of (Freedman, 1963) although it uses the same definitions, essentially the same notation, and improves Remark 6, p. 1400.

**2. Results.** $I$, the population, is a countably infinite set with the discrete topology. $\Lambda$, the set of parameters, is the set of probabilities on $I$, with the weak* topology: $\lambda_n \to \lambda$ if and only if $\lambda_n(i) \to \lambda(i)$ for all $i \, \varepsilon \, I$. Finally, $\pi(\Lambda)$, the set of priors, is the set of probabilities on (the Borel subsets of) $\Lambda$, with the weak* topology: $\mu_n \to \mu$ if and only if $\int_\Lambda f \, d\mu_n \to \int_\Lambda f \, d\mu$ for all bounded, continuous, real functions $f$ on $\Lambda$. It is well known that $f$ can be restricted to polynomials in $\lambda \to \lambda(i): i \, \varepsilon \, I$ without decreasing the topology. If $\lambda \, \varepsilon \, \Lambda$, then $\delta_\lambda \, \varepsilon \, \pi(\Lambda)$ puts mass 1 at $\lambda$. The spaces $\Lambda$ and $\pi(\Lambda)$ are complete, separable, metric.

Consider the coordinate process $\{\xi_n : 1 \leqq n < \infty\}$ on the sample space $I^\infty$ of $I$-sequences with the product topology. If $\lambda \, \varepsilon \, \Lambda$, then $\lambda^\infty$ is the probability on (the Borel subsets of) $I^\infty$ which makes the $\xi_n$ independent with common distribution $\lambda$. If $\mu \, \varepsilon \, \pi(\Lambda)$, the *posterior* $\mu_n$ is this map from part of $I^\infty$ to $\pi(\Lambda)$:

$$\mu_n(d\lambda) = [\,\textstyle\prod_{i=1}^n \lambda(\xi_i)\mu(d\lambda)]/[\int_\Lambda \prod_{i=1}^n \lambda(\xi_i)\mu(d\lambda)],$$

defined when the denominator is positive. The pair $(\lambda, \mu)$ with $\lambda \, \varepsilon \, \Lambda$ and $\mu \, \varepsilon \, \pi(\Lambda)$ is *consistent* if $\lim_{n\to\infty} \mu_n = \delta_\lambda$ with $\lambda^\infty$-probability 1.

A subset of a complete metric space is *first category* if it is a countable union of nowhere dense sets; a subset is *residual* if its complement is first category. For a discussion, see (Kuratowski, 1958, especially Sections 8–11). Topologically, first category sets are small, residual sets large.

The main result of this note is:

THEOREM. *The set $C$ of consistent pairs $(\lambda, \mu)$ is first category in the cartesian product of $\Lambda$ and $\pi(\Lambda)$.*

COROLLARY. *There is a residual set of priors $\mu \, \varepsilon \, \pi(\Lambda)$ with the property: $\{\lambda : \lambda \, \varepsilon \, \Lambda, (\lambda, \mu)$ is consistent$\}$ is first category in $\Lambda$.*

It is convenient to prove a result stronger than the theorem:

PROPOSITION. *The set $R$ of pairs $(\lambda, \mu)$ with $\limsup_{n\to\infty} \int_{I^\infty} \mu_n(U) \, d\lambda^\infty = 1$ simultaneously for all nonempty open subsets $U$ of $\Lambda$ is residual.*