ON THE ASYMPTOTIC POWER OF THE ONE-SAMPLE KOLMOGOROV-SMIRNOV TESTS[1]

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1. Introduction. Let $X_1, X_2, \cdots, X_n$ be a random sample of n observations from some unknown distribution function F, and let

$$\begin{aligned} F_n(x) &= 0, & x &< X_{(1)}, \\ &= i/n, & X_{(i)} &\leqq x < X_{(i+1)}, \\ &= 1, & x &\geqq X_{(n)}, \end{aligned}$$

where $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(n)}$ are the ordered observations. If H is some completely specified continuous distribution function, we may reject the hypothesis that $F = H$ for large values of $K_n = \sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)|$ or $K_n^+ = \sup_{-\infty<x<\infty} n^{\frac{1}{2}}[F_n(x) - H(x)]$; specifically, if $K_n \geqq d_n(\alpha)$ where $P\{K_n \geqq d_n \mid F = H\} = \alpha$ or if $K_n^+ \geqq d_n^+(\alpha)$ where $P\{K_n^+ \geqq d_n^+ \mid F = H\} = \alpha$. These probabilities do not depend on the true underlying distribution function, so long as it is continuous. The test based on K_n was first proposed in 1933 by Kolmogorov [9], and the related K_n^+ test was later suggested by Smirnov [12]; they are called respectively the two-sided and one-sided one-sample Kolmogorov-Smirnov tests of goodness of fit. For a fuller expository treatment we refer to the paper by Darling [5] which also includes an extensive bibliography.

The power of the K_n test when F is equal to some alternative continuous distribution function G is

$$P\{K_n \geqq d_n(\alpha) \mid F = G\} = P\{\sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)| \geqq d_n(\alpha) \mid F = G\},$$

and if $\{G_n\}$ is some sequence of alternative distributions, we may define the asymptotic power against $\{G_n\}$ to be

$$\lim_{n\to\infty} P\{\sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)| \geqq d_n(\alpha) \mid F = G_n\}$$

if this limit exists. Following Doob [7], we introduce the stochastic process $Z_n(t) = n^{\frac{1}{2}}(F_n[F^{-1}(t)] - t)$, $0 \leqq t \leqq 1$; then the asymptotic power may be rewritten in the form

$$\lim_{n\to\infty} P\{\sup_{0<t<1} |Z_n(t) - n^{\frac{1}{2}}(H[G_n^{-1}(t)] - t)| \geqq d_n(\alpha) \mid F = G_n\}.$$

We may omit the condition $F = G_n$ in this expression, since all probability state-

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