ON MOMENT GENERATING FUNCTIONS AND RENEWAL THEORY[1]

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In this paper, which is complementary to [2], we discuss renewal theory for a distribution function having at least one tail that decreases exponentially fast.

Let F denote a one-dimensional right-continuous probability distribution function and f its characteristic function, defined by

$$f(\theta) = \int_{-\infty}^{\infty} e^{i\theta x}\, dF(x), \qquad -\infty < \theta < \infty.$$

Let $F^{(n)}$ denote the n-fold convolution of F with itself and H the renewal function defined by

$$H(x) = \sum_{n=0}^{\infty} F^{(n)}(x), \qquad -\infty < x < \infty.$$

We call F a lattice distribution function with lattice constant $d > 0$ if the measure corresponding to F is concentrated on the set $\{j\, d \mid -\infty < j < \infty\}$ but not on the set $\{j\, d' \mid -\infty < j < \infty\}$ for any $d' > d$. F is lattice with lattice constant d if and only if f is periodic with period $2\pi\, d^{-1}$ and $f(\theta) \neq 1$ for $-\pi\, d^{-1} \leqq \theta \leqq \pi\, d^{-1}$, $\theta \neq 0$. F is non-lattice if and only if $f(\theta) \neq 1$ for $\theta \neq 0$. As a special case, we call F strongly non-lattice if

$$\liminf_{|\theta| \to \infty} |1 - f(\theta)| > 0.$$

In the non-lattice case set $d = 0$. In general, set $[x]_d = d[x/d]$ for $d > 0$ and $[x]_d = x$ for $d = 0$.

Theorem. *Let F have finite first moment $\mu > 0$. (i) If for some $r_1 > 0$, $F(x) = o(e^{r_1 x})$ as $x \to -\infty$, then for some $r > 0$*

$$(1) \qquad H(x) = o(e^{rx}) \qquad \text{as } x \to -\infty.$$

(ii) *If F has finite second moment μ_2, if for some $r_1 > 0$, $1 - F(x) = o(e^{-r_1 x})$ as $x \to \infty$, and if F is either lattice or strongly non-lattice, then for some $r > 0$*

$$(2) \qquad H(x) = \{[x]_d + (d/2)\}/\mu + \mu_2/2\mu^2 + o(e^{-rx}) \qquad \text{as } x \to \infty.$$

Note that $[x]_d + (d/2) = x$ in the non-lattice case. Actually, in (i) we can allow $\mu = +\infty$ with no essential change in proof. The above theorem was suggested by [2] and a remark of Gelfond [1].

Proof. Let g denote the moment generating function of F defined by

$$g(s) = \int_{-\infty}^{\infty} e^{sx}\, dF(x),$$

the domain being all complex numbers s for which the integral exists absolutely. Then $g(i\theta) = f(\theta)$, $-\infty < \theta < \infty$.

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