ON STOCHASTIC PROCESSES DERIVED FROM MARKOV CHAINS[1]

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0. Introduction. If $X_0, X_1, \cdots$ is a stochastic process with finite state space S, T is a finite set and $f: S \to T$ then $fX_0, fX_1, \cdots$ is a stochastic process with state space T, often referred to as a "functional" of $X_0, X_1, \cdots$. Our object here is to characterize processes which are functionals of Markov chains.

This problem was considered by E. J. Gilbert [4] who adduced a necessary condition. S. W. Dharmadikari [1], [2], [3] showed that Gilbert's condition was not sufficient and provided a sufficient condition but, though he has given an excellent analysis of the problem, did not complete the characterization. We do this here (Theorem 5.1).

Dharmadikari has indicated the essentially geometrical nature of the problem. With a stochastic process he associates vector spaces on which the states operate linearly. His conditions (as well as ours) have to do with invariant convex cones in these spaces. We have chosen here to regard such spaces as modules over the free associative algebra generated by the state space S (i.e. the algebra of polynomials in the noncommuting variables $x \varepsilon S$). This point of view, still unconventional perhaps in probability theory, seems indicated by the fact that the vector spaces in question lack preferred bases, so that linear transformations are not naturally represented by matrices. We believe the argument is made simpler and more conceptual by its adoption.

To each stochastic process with finite state space we associate canonically a module; the process is then discussed in terms of this module. If the module is finite dimensional (this is Gilbert's necessary condition) it is reasonable to say that the process is characterized by finitely many parameters: this condition obviously defines an interesting class of processes. We suggest that the proper apparatus for the discussion of such processes is the one introduced here.

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1. Stochastic S-modules. We shall be concerned here with stochastic processes with discrete time and finite state space; "stochastic process" is used below in this sense only. Such a process, with state space S, is completely specified by the probabilities $p(x_1, \cdots, x_n)$ of the finite sequences $x_1, \cdots, x_n$ in S.

It is more convenient for our purposes to describe these processes in the following way. Let A_S be the free associative $\mathbf{R}$-algebra generated by S. We then

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