LIMIT THEOREMS FOR QUEUES WITH TRAFFIC INTENSITY ONE[1]

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1. Introduction. We shall consider a single server queueing process in which customers are served in the order of their arrival. Let the instants customers arrive at the counter be denoted by $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. (We shall follow, for the most part, the notation used in Takács [14].) Let $\theta_n = \tau_{n+1} - \tau_n$ ($n = 0, 1, \cdots; \tau_0 = 0$) denote the interarrival times and let χ_n be the service time of the nth customer. We shall assume that $\{\theta_n\}$ and $\{\chi_n\}$ are independent sequences of independent, identically distributed, positive random variables. In particular, we shall assume that θ_n has an exponential distribution with mean $1/\lambda$ (the input process is a homogeneous Poisson process with intensity λ) and that χ_n has an arbitrary distribution, H. This queueing process is often designated $M/G/1$.

There are three important characteristics of the queue which we shall be interested in studying. These characteristics are the queue size (number of customers in the system), the busy period (interval of time during which the server is busy), and the waiting time of the customers. Let $\xi(t)$ denote the number of customers in the system at time t; i.e., the number of customers waiting or being served at time t. Let $\eta(t)$, the so-called virtual waiting time at time t, be the time that a customer would wait before starting his service, if he joined the queue at time t. We shall let $\tau_1', \tau_2', \cdots, \tau_n', \cdots$ denote the instants at which customers complete their service and depart from the system. Then if we let $\xi_n \equiv \xi(\tau_n' + 0)$, ξ_n will be the number of customers left in the system at the moment the nth customer departs. Also, if we define $\eta_n \equiv \eta(\tau_n - 0)$, η_n will be the time the nth customer waits before starting his service.

In the study of queueing processes an important role is played by the traffic intensity ρ, which is defined as the ratio of the expected service time to the expected interarrival time. If we denote the mean of the service distribution by μ, then $\rho \equiv \lambda\mu$. If $\rho < 1$, it has been shown by Takács [14] and others that the stochastic processes $\{\eta(t): t \geqq 0\}$, $\{\eta_n : n \geqq 1\}$, $\{\xi(t): t \geqq 0\}$, and $\{\xi_n : n \geqq 1\}$ converge in distribution to non-degenerate limiting distributions. In this case it is usually said that the queueing process attains a steady-state. The great bulk of papers in the queueing literature deal exclusively with the steady-state situation. On the other hand, if $\rho \geqq 1$, the distributions of the processes mentioned above tend to zero as either n or t tend to infinity.

In this paper we shall treat exclusively the case $\rho = 1$. Without loss of generality we shall choose our unit of time so that $\lambda = \mu = 1$. We shall be interested in obtaining limit distributions for certain functionals of the stochastic processes mentioned in the last paragraph. The basic tool used in obtaining these limit

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