CONDITIONAL EXPECTATION GIVEN A σ-LATTICE AND APPLICATIONS

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1. Introduction. The applications discussed are applications of an extremizing property of the conditional expectation given a σ-lattice. It has been interesting to the author to observe that a variety of problems of maximum likelihood estimation of parameters, of functions, of probability densities, lead to a class of extremum problems of which the conditional expectation given a σ-lattice furnishes the solution. By a σ-lattice is understood a family of sets closed under countable union and countable intersection, but not necessarily difference or complement. In extending in a natural way to σ-lattices [11] the concept of conditional expectation given a σ-field, one sacrifices a very useful property indeed, linearity. On the other hand, much carries over; in particular, some theorems of martingale type are proved in Sections 2 and 3.

2. Projection on a closed convex set in a complete inner product space. Let H be a complete inner product space: a complete metric space with distance between elements X, Y given by $\|X - Y\|$, where $\|Z\| =_D (Z, Z)^{\frac{1}{2}}$ for $Z \varepsilon H$, and where $(\cdot, \cdot)$ denotes the inner product. If $Y \varepsilon H$, $Z \varepsilon H$, then seg YZ will denote the segment joining Y and Z; i.e., seg $YZ =_D \{W : W \varepsilon H, \exists \alpha \ni 0 \leqq \alpha \leqq 1 \text{ and } W = (1 - \alpha)Y + \alpha Z\}$. A subset A of H will be termed closed if it is closed in the topology of the metric above. The set A is convex if $Y \varepsilon A, Z \varepsilon A \Rightarrow$ seg YZ $\subset A$.

The following existence theorem is well known.

Theorem 2.1. *Let A be a closed convex set in H, and let $X \varepsilon H$. Then there is a unique closest point Y of A to X.* For let $\{Y_n\}$ be a sequence of points of A whose distances from X approach the infimum of such distances, and set $U_n = Y_n - X$. From the fact that $(Y_n + Y_m)/2 \varepsilon A$ for all n and m and from the equation

$$\|U_n - U_m\|^2 = 2(\|U_n\|^2 + \|U_m\|^2) - \|U_n + U_m\|^2$$

it follows that $\{Y_n\}$ is Cauchy and hence has a limit $Y \varepsilon A$. Uniqueness also follows immediately from the convexity of A and the identity

$$\|(Y_1 + Y_2)/2 - X\|^2 = (\|Y_1 - X\|^2 + \|Y_2 - X\|^2)/2 - \|Y_1 - Y_2\|^2/4.$$

The notation $P(X \mid A)$ will be used for the closest point of a closed convex set A to the point X.

Theorem 2.2. *If $X \varepsilon H$ and if A is a closed convex subset of H, then $Y = P(X \mid A)$ if and only if*

$$(X - Y, Y - Z) \geqq 0 \quad \textit{for all} \quad Z \varepsilon A. \tag{2.1}$$

Proof. This is immediate from Theorem 2.1 and the observation that if

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