PEAKEDNESS OF DISTRIBUTIONS OF CONVEX COMBINATIONS[1]

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1. Introduction. Roughly speaking, the law of large numbers states that under mild restrictions the average of a random sample has small probability of deviating from the population mean if the sample size n is taken large enough. However, nothing is said about the probability of a given size deviation decreasing monotonically as n increases. In this paper we develop conditions under which such monotonicity can be established. Another way of stating this is that under appropriate conditions the "peakedness" of the distribution of the average of n increases with n. We use the definition of peakedness given by Birnbaum (1948).

Definition. Let X_1 and X_2 be real random variables and a_1 and a_2 real constants. We say X_1 is more peaked about a_1 than X_2 about a_2 if

$$P[|X_1 - a_1| \geqq t] \leqq P[|X_2 - a_2| \geqq t] \tag{1.1}$$

for all $t \geqq 0$. In the case $a_1 = 0 = a_2$, we shall simply say X_1 is more peaked than X_2.

If the inequality between the two probabilities in (1.1) is strict whenever the two probabilities are not both 0, we say X_1 is strictly more peaked about a_1 than X_2 about a_2.

2. Peakedness comparisons for symmetric Pólya frequency functions of order 2.

Lemma 2.1. *Let f be a Pólya frequency function of order* 2 (PF_2), $f(u) = f(-u)$ *for all u, X_1 and X_2 independently distributed with density f. Then $pX_1 + qX_2$ is strictly increasing in peakedness as p increases from* 0 *to* $\frac{1}{2}$, *with* $p + q = 1$.

Proof. For $0 < p < \frac{1}{2}$, define

$$G_2(p, t) = P[pX_1 + qX_2 \leqq t] = \int_{-\infty}^{\infty} F((t - qu)/p)f(u)\, du.$$

Then $p^2(\partial G_2/\partial p) = \int_{-\infty}^{\infty} f((t - qu)/p)f(u)(u - t)\, du$; differentiation under the integral sign is permissible since $|f((t - qu)/p)f(u)(u - t)| \leqq Mf(u)|u - t|$ and $\int_{-\infty}^{\infty} Mf(u)(u - t)\, du < \infty$, where M is the modal ordinate of f. Rewrite

$$p^2(\partial G_2/\partial p) = \int_{-\infty}^{t} f((t - qu)/p)f(u)(u - t)\, du + \int_{t}^{\infty} f((t - qu)/p)f(u)(u - t)\, du.$$

Let $v = t - u$ in the first integral and $v = u - t$ in the second integral. We get

$$p^2(\partial G_2/\partial p) = \int_0^{\infty} v\{f(t + v)f(t - (qv/p)) - f(t - v)f(t + (qv/p))\}\, dv. \tag{1}$$

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