ON ABSOLUTELY CONTINUOUS COMPONENTS AND RENEWAL THEORY[1]

By Charles Stone

University of California, Los Angeles

1. Introduction. In this paper we continue the work of [6] and [7] by studying renewal theory in the case of an absolutely continuous component.

Let μ be a probability measure on the Borel subsets of the reals R, let $\mu^{(n)}$ denote the n-fold convolution of μ with itself and set $\nu = \sum_0^\infty \mu^{(n)}$. We assume throughout this paper that *some* $\mu^{(n)}$ *has a non-trivial absolutely continuous component*. If ν assigns measure to bounded intervals, this is equivalent to the assumption that ν has a non-trivial absolutely continuous component. Our main goal is to write $\nu = \nu' + \nu''$, where ν' is absolutely continuous and its density has the proper tail behavior for a renewal density, and ν'' is a finite measure and has the same tail behavior as μ.

To this end we can find an n_0 and probability measures φ and φ' such that $\mu^{(n_0)} = (\varphi + \varphi')/2$, φ has compact support, is absolutely continuous and has a twice continuously differentiable density. Set $\nu''' = \sum_0^\infty \mu^{(nn_0)}$. Let $\hat{\mu}$ denote the characteristic function of μ and let the characteristic functions of the other finite measures be denoted similarly. Also set $\hat{\nu} = (1 - \hat{\mu})^{-1}$ and $\hat{\nu}''' = (1 - \hat{\mu}^{n_0})^{-1}$.

Now $\hat{\nu}''' = 1 + \hat{\mu}^{n_0}\hat{\nu}'''$ and $\hat{\mu}^{n_0} = (\hat{\varphi} + \hat{\varphi}')/2$, from which it follows that

$$\hat{\nu}''' = (1 - \tfrac{1}{2}\hat{\varphi}')^{-1}(1 + \tfrac{1}{2}\hat{\varphi}\hat{\nu}''').$$

Thus

$$\begin{aligned}\hat{\nu} &= \textstyle\sum_{n=0}^\infty \hat{\mu}^n = (1 + \hat{\mu} + \cdots + \hat{\mu}^{n_0 - 1}) \sum_{n=0}^\infty \hat{\mu}^{nn_0} \\ &= (1 + \hat{\mu} + \cdots + \hat{\mu}^{n_0-1})(1 - \tfrac{1}{2}\hat{\varphi}')^{-1}(1 + \tfrac{1}{2}\hat{\varphi}\hat{\nu}''').\end{aligned}$$

Correspondingly, $\nu = \nu' + \nu''$, where

$$\nu'' = (1 + \mu + \cdots + \mu^{(n_0-1)}) * \left(\textstyle\sum_{n=0}^\infty (\varphi')^n/2^n\right)$$

and $\nu' = \frac{1}{2}\varphi * \nu'' * \nu'''$, $*$ denoting convolution. Note that ν'' has total measure $2n_0$ and essentially the same tail behavior as μ (to be made more precise below), and that ν' is absolutely continuous and has a continuous density p.

Suppose that μ has a finite positive first moment $\mu_1 = \int x\mu(dx)$. Then $\mu^{(n_0)}$ has first moment $n_0\mu_1$ and the renewal theorem of Blackwell [1] asserts that if I is a bounded interval with length $|I|$, then $\nu'''(I)$ is finite,

$$\lim_{x\to-\infty} \nu'''(x + I) = 0 \quad\text{and}\quad \lim_{x\to+\infty} \nu'''(x + I) = |I|/n_0\mu_1 .$$

Since φ has compact support and a continuous density, it follows easily that

$$\lim_{x\to-\infty} p(x) = 0 \quad\text{and}\quad \lim_{x\to+\infty} p(x) = 1/\mu_1 .$$

Received 22 June 1965.

[1] The preparation of this paper was sponsored in part by N.S.F. Grant GP-1606.