EQUIVALENCE AND SINGULARITY FOR FRIEDMAN URNS[1]

By David A. Freedman

University of California, Berkeley

1. Introduction. W_0 {respectively, W_0'} and B_0 {B_0'} are positive real numbers, α {α'} and β {β'} non-negative real numbers with $\alpha + \beta > 0$ {$\alpha' + \beta' > 0$}. At time 0, urn U {U'} contains W_0 {W_0'} white and B_0 {B_0'} black balls. At time n, a ball is drawn at random from U {U'} and replaced, together with α {α'} balls of the same color and β {β'} of the opposite. If the nth draw from U {U'} is white, X_n {X_n'} is 1; otherwise, 0. The distribution of $X_1, X_2, \cdots$ {$X_1', X_2', \cdots$} is D {D'}, a probability on the space Ω of sequences of 0'r and 1's. Let $\rho = (\alpha - \beta)/(\alpha + \beta)$ {$\rho' = (\alpha' - \beta')/(\alpha' + \beta')$}. The object of this note is to prove

(1) Theorem. *$D \equiv D'$ or $D \perp D'$ according as $\rho = \rho'$ or $\rho \neq \rho'$.*

If $\rho = \rho' = 1$, then (1) follows from De Finetti's theorem; if $\rho < \rho' = 1$, then (1) follows from Reference [2], Lemma 2.1 and Theorems 2.2, 3.1.

2. Generalities. Let $\mathfrak{F}_n$ be the σ-field of subsets of Ω spanned by the first n coordinates. If Π is a probability on Ω, let $\Pi(n + 1, i)$ be the conditional Π-probability that the $n + 1$st coordinate is i, given $\mathfrak{F}_n$. If $\omega \varepsilon \Omega$, let

(2) $$S_n(\omega) = \omega(1) + \cdots + \omega(n)$$

and

(3) $$E_n = n^{-1}S_n - \tfrac{1}{2}.$$

If $\rho < 1$, by [2],

(4) $$E_n \to 0 \quad \text{with } D\text{-probability } 1.$$

Since

(5) $$D(n + 1, 1) = [W_0 + \beta n + (\alpha - \beta)S_n]/[W_0 + B_0 + (\alpha + \beta)n],$$

it follows from (4) that when $\rho < 1$,

(6) $$D(n, 1) \to \tfrac{1}{2} \quad \text{with } D\text{-probability } 1.$$

This may help to motivate the next result.

Let P {respectively, P'} be the probability on the two-point set {0, 1} assigning measure p {p'} to 1. Let $\epsilon = p' - p$.

(7) Lemma. *If p and p' converge to $\frac{1}{2}$, then the P-expectation of $\log (dP'/dP)$ is $-2\epsilon^2 + o(\epsilon^2)$, and the P-expectation of $(\log (dP'/dP))^2$ is $4\epsilon^2 + o(\epsilon^2)$.*

Proof. Expand $\log (1 + u)$ in powers of u. ∎

Received 19 July 1965; revised 30 August 1965.

[1] Partially supported by the National Science Foundation, Grant GP-2593; and by the Sloan Foundation.