ON THE BIVARIATE MOMENTS OF ORDER STATISTICS FROM A LOGISTIC DISTRIBUTION[1]

BY B. K. SHAH

Yale University and Purdue University

1. Introduction. A logistic distribution is defined by $x = \ln \{F/(1 - F)\}$, where F is the probability of a value less than x. This is a symmetric distribution with mean zero and variance $\pi^2/3$. The shape of a logistic distribution is nearly the same as that of a normal distribution except at the tails. Birnbaum [2], Birnbaum and Dudman [3], Plackett [9] and others [8] have given tables of the expected values of the order statistics.

In Section 3 a convenient expression for the moment generating function of the ith and jth order statistic ($j > i$) in a random sample of size n drawn from a logistic distribution is derived. This expression is useful in deriving the higher product moments of the order statistics. In Section 4, a finite and easily computable expression [11] is developed. Also various recurrence relations are obtained. In Section 5, using digamma and trigamma values tabulated in [5], [6], we give the covariances of all pairs of order statistics up to sample size $n = 10$.

2. Notation and order statistics theory. Let $x_{1,n} \leqq x_{2,n} \leqq \cdots x_{n,n}$ be the order statistics in a sample of size n from any continuous distribution. Let the cumulative distribution function (c.d.f.) be denoted by $F(x)$. It is well known that the distribution of ith order statistics has the probability differential element

$$(2.1) \qquad a_{i,n}(x)\, dx = [B(i, n - i + 1)]^{-1} F^{i-1}(x)[1 - F(x)]^{n-i}\, dF(x), \qquad i = 1, 2, \cdots, n,$$

where $B(k, m) = \Gamma(k)\Gamma(m)/\Gamma(k + m)$, $k > 0$, $m > 0$. And the joint distribution of ith and jth order statistics is

$$(2.2) \qquad \begin{aligned} a_{i,j,n}(x, y)\, dx\, dy = {} & [B(i, j - i)B(j, n - j + 1)]^{-1} F^{i-1}(x) \\ & \cdot [F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j}\, dF(x)\, dF(y), \end{aligned} \qquad x < y \quad \text{and} \quad 1 \leqq i < j \leqq n.$$

Let

$$(2.3) \qquad \mu_{i,n}^{(k)} = E(X_{i,n}^k) = \int_{-\infty}^{\infty} x^k a_{i,n}(x)\, dx, \quad 1 \leqq i \leqq n, \quad k = 1, 2, \cdots,$$

with $\mu_{i,n}^{(1)} = \mu_{i,n}$ and

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