CONVERGENCE PROPERTIES OF CONVERGENCE WITH PROBABILITY ONE

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Convergence in probability, convergence in the sense of Levy or Prohorov, and other familiar types of stochastic convergence are topological in nature. For countably additive probabilities, Thomasian (1957) gives necessary and sufficient conditions that convergence with probability one be equivalent to convergence in a metric or a norm defined on the space of random variables. In this paper we discuss when convergence with probability one is a topological convergence for finitely or countably additive probabilities. For finitely additive probabilities we employ the decomposition theorem of Yosida and Hewitt (1952).

Let $(\mathfrak{X}, \mathfrak{A}, \gamma)$ be a finitely additive probability space: γ is finitely but not necessarily countably additive and $\mathfrak{A}$ is a sigma-field (see Birkhoff, 1948, p. 185). Let $\mathfrak{Y}$ be a nontrivial normed linear space with norm $\|\cdot\|$. Let $\mathfrak{M}$ be a vector space, with the same scalars as $\mathfrak{Y}$, of functions defined on $\mathfrak{X}$ with values in $\mathfrak{Y}$ which contain indicator functions of sets: there is $y \neq 0$ in $\mathfrak{Y}$ so that the function $I_A = 0$sr y as $x \varepsilon A^c$ or $x \varepsilon A$ belongs to $\mathfrak{M}$.

Definitions. A topology $\mathfrak{I}$ for $\mathfrak{M}$ is equivalent to convergence with probability one if for each net $\{f_d : d \varepsilon D\}$ in $\mathfrak{M}$, $\lim_d f_d = f \varepsilon \mathfrak{M}$ relative to $\mathfrak{I}$ if and only if there is $M \varepsilon \mathfrak{A}$ with $\gamma(M) = 0$ and $\lim_d \|f_d(x) - f(x)\| = 0$ for $x \varepsilon M^c$ (we denote the latter by $\gamma(\lim_d f_d = f) = 1$). A topology $\mathfrak{I}$ for $\mathfrak{M}$ is equivalent to sequential convergence with probability one if we replace "net" by "sequence" in the foregoing definition.

We rephrase in probabilistic terminology and slightly specialize in (2), below, two *necessary* conditions [Kelley (1955) p. 69 and 74] for a $\mathfrak{I}$ to be equivalent to convergence with probability one:

(1) If it is false that $\gamma(\lim_d f_d = f) = 1$ then there is a subnet $\{g_e : e \varepsilon E\}$ of $\{f_d\}$ whose domain is a cofinal subset of D and which has no subnet which converges with probability one to f.

(2) Let $\mathfrak{C}$ be a directed set and for each $C \varepsilon \mathfrak{C}$ let $N_C \equiv N$, the nonnegative integers. If $\gamma(\lim_C f_C = f) = 1$ and if $\gamma(\lim_n g_{C,n} = f_C) = 1$, $C \varepsilon \mathfrak{C}$, then there is a directed set D and a function $R: D \to \mathfrak{C} \times N$ so that $\gamma(\lim_d g \circ R = f) = 1 (g(C, n) \equiv g_{C,n})$.

Here are some propositions which help characterize when γ, finitely or countably additive, may fail to satisfy (1) and/or (2). Recall the following definitions: (i) γ is purely finitely additive if the only countably additive nonnegative measure φ on satisfying $\varphi(A) \leqq \gamma(A)$, $A \varepsilon \mathfrak{A}$ is $\varphi \equiv 0$ [Yosida and Hewitt, (1952)]; (ii) $A \varepsilon \mathfrak{A}$ is a positive γ-atom if $\gamma(A) > 0$ and for each $B \subset A$, $B \varepsilon \mathfrak{A}$ either $\gamma(B) = 0$ or $\gamma(B) = \gamma(A)$.

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