A NOTE ON RISK AND MAXIMAL REGULAR GENERALIZED SUBMARTINGALES IN STOPPING PROBLEMS[1]

By H. Chernoff

Stanford University

In a recent paper Haggstrom [2] rearranged and amplified on the excellent work of Snell [3] on optimal stopping problems as a preliminary to generalizing the framework to deal with problems in control and experimental design. The purpose of this note is to point out that one of Haggstrom's results can be derived under weaker conditions by using the results of Snell.

Let $\{Z_n, F_n, n \geqq 1\}$ be a stochastic process on a probability space (Ω, F, P) with points ω. A stopping variable (sv) is a random variable (rv) t with values in $\{1, 2, \cdots, \infty\}$ such that $t < \infty$ a.e. and $\{t = n\} \varepsilon F_n$ for each n. For any such sv t, a rv Z_t is defined by $Z_t(\omega) = Z_n(\omega)$ if $t(\omega) = n$ and $Z_t(\omega) = \infty$ if $t(\omega) = \infty$. The optimal stopping problem consists of finding t to minimize the risk $E(Z_t)$.

Random variables are defined in the extended sense, i.e., they can take on the values $+\infty$ and $-\infty$. We regard $E(X)$ as defined as long as either the positive or negative part of X has finite expectation. This extension is usually referred to by the term generalized. The reader is referred to Snell [3] or Haggstrom [2] for detailed discussion of the technical terms used in this note.

Snell [2] obtained results characterizing the solution of the optimal stopping problem in terms of $\{Y_n, F_n, n \geqq 1\}$, the maximal regular generalized submartingale relative to $\{Z_n, F_n, n \geqq 1\}$. In particular, as part of his Theorem 3.6, he proved that

$$E(Y_n) = \inf_{t \varepsilon T_n} E(Z_t) \tag{1}$$

where T_n is the class of sv for which $t \geqq n$ a.e. The proof of this theorem used the hypothesis that

$$E(\inf_n Z_n) > -\infty. \tag{2}$$

Haggstrom [1] introduced, and developed his results in terms of,

$$X_n = \operatorname{ess\,inf}_{t \varepsilon T_n} E(Z_t \mid F_n) \tag{3}$$

which represents the optimal risk at stage n to the player who, for one reason or another, has not stopped previously. The relation between the results of Snell and Haggstrom, and in fact the motivation of Snell, is clarified by Haggstrom's Theorem 3.5 which states,

$$X_n = Y_n \quad \text{a.e.} \qquad n = 1, 2, \cdots \tag{4}$$

Received 26 September 1966.

[1] This paper was prepared with the partial support of the National Science Foundation Grant GP-5705.