ON AN INEQUALITY OF HOEFFDING

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1. Introduction. Let $X_1, X_2, \cdots, X_n$ be a sample drawn without replacement from the finite population π, and let $Y_1, Y_2, \cdots, Y_n$ be a sample drawn with replacement from the same population. We set

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{and} \quad Z_n = Y_1 + Y_2 + \cdots + Y_n .$$

The distribution of S_n is, to a greater extent concentrated at its mean than is the distribution of Z_n. A quantitative formulation of this fact is, for example, the well-known relation

$$\sigma^2(S_n) \leqq \sigma^2(Z_n) \tag{1.1}$$

where σ^2 denotes variance.

The following theorem, due to Hoeffding (Theorem 4 in [3]), is a considerably more informative result in this direction.

Theorem. *For any convex and continuous function $\varphi(x)$ we have*

$$E\varphi(S_n) \leqq E\varphi(Z_n). \tag{1.2}$$

(E denotes here, and in the sequel, mathematical expectation.)

If we, for example, choose $\varphi(x) = (x - n\mu_\pi)^2$, where μ_π is the mean in the population π, then (1.2) becomes just (1.1).

The purpose of this paper is to generalize Hoeffding's result in two directions, one of which consists in showing that (1.2) holds for certain types of sample functions other than convex continuous functions of the sample sum. The other direction of generalization is to prove that (1.2) holds not only when $Y_1, Y_2, \cdots, Y_n$ are sampled with replacement, but for a broader class of sampling procedures, here called symmetric sampling procedures. Loosely speaking, a sampling procedure is said to be symmetric if all elements in the population are treated symmetrically during the drawing procedure. So for example sampling with replacement every second time is symmetric while sampling with probability proportional to size is not. In Section 2 we formally define sampling procedures and in particular symmetric procedures. In recent years problems in survey sampling have inspired various formalizations of the concept of sampling procedure, see Godambe [1], of which our formalization is a special case.

2. About sampling in general. Let (m) stand for the set of the first m integers, i.e. $(m) = (1, 2, \cdots, m)$ and let $\Omega(n, N) = (N) \times (N) \times \cdots \times (N)$ (n factors, $\times$ denotes Cartesian product). Points in $\Omega(n, N)$ will be denoted by $(i_1, i_2, \cdots, i_n)$ or by ω.

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