AN OSCILLATING SEMIGROUP[1]

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1. Introduction. Let I be a countably infinite set. For each $t \geqq 0$, let $P(t) = \{P(t, i, j)\}$ be a stochastic matrix on I, such that $P(t + s) = P(t)P(s)$, $P(0)$ is the identity matrix, and $P(t) \to P(0)$ coordinatewise at $t \to 0$. Then P is called a standard stochastic semigroup on I. As is well known, P has a coordinatewise derivative Q at 0. However, many elements of Q may vanish. In view of this, L. E. Dubins asked me whether $P(t, i, j)/P(t, i, k)$ converged as $t \to 0$. The object of this note is to provide a counterexample.

(1) Theorem. *There is a countable set I, with elements 0, 1, 2 and a standard stochastic semigroup P on I, satisfying*

$$\limsup_{t\to 0} P(t, 0, 1)/P(t, 0, 2) = \infty \tag{2}$$

$$\liminf_{t\to 0} P(t, 0, 1)/P(t, 0, 2) = 0.$$

Moreover, there is a Markov chain with stationary transitions P, starting from 0, all of whose sample functions are step functions. In particular, all elements of Q are finite.

The construction is given in Section 2, two preliminary facts in Section 3, and the verification in Section 4. Section 5 contains some technical remarks.

2. Construction. The state space I consists of 0, 1, 2, $(1, n, m)$ and $(2, n, m)$ for positive integer n and $m = 1, \cdots, f(n)$. Here $f(n)$ is positive integer to be chosen later. Think of it as large.

Let $0 < q_{n,m} < \infty$, and let

$$c_n = \sum_{m=1}^{f(n)} q_{n,m}^{-1}. \tag{3}$$

The $q_{n,m}$ will be chosen later. Think of them as very large. Let $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} (a_n + b_n) = 1$,

$$a_{n+1} + a_{n+2} + \cdots = o(a_n), \tag{4}$$

$$b_{n+1} + b_{n+2} + \cdots = o(b_n),$$

$$\limsup_{n\to\infty} a_n/b_n = \infty, \qquad \liminf_{n\to\infty} a_n/b_n = 0. \tag{5}$$

Let τ_0 be exponential with parameter 1, and $\tau_{n,m}$ exponential with parameter $q_{n,m}$, for $n = 1, 2, \cdots, m = 1, \cdots, f(n)$, all independent. In particular,

$$P(\tau_{n,m} \geqq t) = e^{-q_{n,m}t} \quad \text{for} \quad t \geqq 0.$$

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