SERIES REPRESENTATIONS OF DISTRIBUTIONS OF QUADRATIC FORMS IN NORMAL VARIABLES II. NON-CENTRAL CASE

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This paper is a direct continuation of Part I. [5] Equation numbers continue from those of the earlier paper.

We recall that

(i) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ is the probability density function of $\sum_{i=1}^{n} \alpha_i(Z_i + \delta_i)^2$ where $Z_1, Z_2, \cdots, Z_n$ are mutually independent standardized normal variables, and $F_n(\boldsymbol{\alpha}; \delta; y)$ is the corresponding cumulative distribution function.

(ii) $f(n; \boldsymbol{\lambda}; y) = f_n(\mathbf{1}; \boldsymbol{\delta}; y)$ with $\boldsymbol{\lambda} = \sum_{i=1}^{n} \delta_i^2$ is the non-central χ^2 probability density function with n degrees of freedom and non-centrality parameter $\boldsymbol{\lambda}$, and $F(n; \boldsymbol{\lambda}; y)$ the corresponding cumulative distribution function.

(iii) $g(n; y) = f(n; 0; y)$ is the central χ^2 probability density function, and $G(n; y)$ the corresponding cumulative distribution function.

In this paper we seek expansion for $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ in each of the following forms:

(101) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^P(-1)^k(y/2)^{n/2+k-1}/2\Gamma(n/2 + k)$ (*Power Series*).

(102) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^L g(n; y/\beta)[k!\,\Gamma(n/2)/\beta\Gamma(n/2 + k)]L_k^{(n/2-1)}(y/2\beta)$

(*Laguerre Series*).

(See (19) and (20) for definition of $L_k^{(n/2-1)}$.)

(103) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^c\beta^{-1}g(n + 2k; y/\beta)$ (*Chi-squared Series*).

(104) $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k^{c'}\beta^{-1}f(n + 2k; \boldsymbol{\lambda}; y/\beta)$

(*Non-central Chi-squared Series*).

We now restrict ourselves to the positive definite forms with $\alpha_r > 0$. The general methods used are almost identical to those described in the introduction to Part I with $g_n(\boldsymbol{\alpha}; y)$ replaced by $f_n(\boldsymbol{\alpha}; \delta; y)$. Briefly, we first find the Laplace transform of $f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$ which is

(105)
$$L_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; s) = \int_0^\infty e^{-sy} f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)\, dy$$
$$= \exp\left(-\sum_{r=1}^{n} \delta_r^2\alpha_r^s/(1 + 2s\alpha_r)\right)\cdot\prod_{j=1}^{n}(1 + 2s\alpha_j)^{-\frac{1}{2}}.$$

We seek an expansion

(106)
$$f_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y) = \sum_{k=0}^{\infty} a_k h_k(y),$$

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