SERIES REPRESENTATIONS OF DISTRIBUTIONS OF QUADRATIC FORMS IN NORMAL VARIABLES. I. CENTRAL CASE

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1. Introduction. Suppose that $X' = (X_1, X_2, \cdots, X_n)$ is a random vector with a multivariate normal distribution with expected value vector $(\xi_1, \xi_2, \cdots, \xi_n)$ and variance-covariance matrix V.

The object of this paper is to give unified derivations of a number of series representations of the distributions of quadratic forms

$$Q(X) = Q(X_1, \cdots, X_n) = X'AX = \sum_{i=1}^n \sum_{j=1}^n a_{ij}X_iX_j$$

where A is a real symmetric matrix.

The representations are all known, but the method of derivation presented here differs from earlier methods, sometimes slightly, sometimes substantially. We also give recurrence relationships for calculating coefficients of the series which are, in some cases, simpler than those heretofore available. Further we obtain a number of new bounds for errors committed in truncating the various series.

The cumulative distribution function of $Q(X)$ is

$$\Pr [Q(X) \leqq y]. \tag{1}$$

By performing suitable linear transformations, (see, e.g. [9]), (1) can be shown to be equal to

$$\Pr [\textstyle\sum_{i=1}^n \alpha_i(Z_i + \delta_i)^2 \leqq y] \tag{2}$$

where Z_i's are independent unit normal variables, and $\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \cdots \geqq \alpha_n$ are the eigenvalues of VA. The δ_i's are the same functions of the ξ_i's as the Z_i's are of the X_i's.

In the special (central or homogeneous) case when $\xi_1 = \xi_2 = \cdots = \xi_n = 0$, all δ_i's are also equal to zero and (2) becomes:

$$\Pr [\textstyle\sum_{i=1}^n \alpha_i Z_i^2 \leqq y]. \tag{3}$$

We will denote the cumulative distribution functions (2) and (3) by

$$F_n(\alpha_1, \alpha_2, \cdots, \alpha_n; \delta_1, \delta_2, \cdots, \delta_n; y) \equiv F_n(\boldsymbol{\alpha}; \boldsymbol{\delta}; y)$$

and $G_n(\boldsymbol{\alpha}, y)$ respectively. The corresponding densities will be denoted by lower case letters. Thus $g_n(1, 1, \cdots, 1; y) = g_n(\mathbf{1}; y)$ is the probability density of a central χ^2 with n degrees of freedom; while $f_n(\mathbf{1}; \boldsymbol{\alpha}; y)$ corresponds to a noncentral

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