ON THE GLIVENKO-CANTELLI THEOREM FOR INFINITE INVARIANT MEASURES[1]

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1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a sigma-finite measure space. Let τ be a (i) *measure preserving* (ii) *conservative* (iii) *ergodic* point-transformation on Ω. That is, we assume that: (i) $A \varepsilon \mathcal{A}$ implies $\tau^{-1}(A) \varepsilon \mathcal{A}$ and $\mu(\tau^{-1}A) = \mu(A)$; (ii) $A \varepsilon \mathcal{A}$, $A \cap \tau^{-i}A = \emptyset$ for $i = 1, 2, \cdots$ implies $\mu(A) = 0$; (iii) the *invariant sigma-field* $\mathcal{I} = \{A : \tau^{-1}A = A \varepsilon \mathcal{A}\}$ is trivial, i.e. $A \varepsilon \mathcal{I}$ implies $\mu(A) = 0$ or $\mu(\Omega - A) = 0$. In probability theory, null-recurrent Markov chains and Markov processes satisfying the Harris condition give rise to such transformations (see Harris and Robbins [4], Harris [3], Kakutani and Parry [6]).

Let X_0, Y_0 be fixed real-valued measurable functions on Ω and let $X_n = X_0 \circ \tau^n$, $Y_r = Y_0 \circ \tau^n$, $n = 1, 2, \cdots$. If s, x, t, y are extended real numbers, let

$$F_n^s(x) = 1_{(s,x)} \circ X_n, \qquad G_n^t(y) = 1_{(t,y)} \circ Y_n, \qquad n = 0, 1, \cdots, \tag{1.1}$$

and

$$F^s(x) = \int_\Omega F_0^s(x)\mu(d\omega), \qquad G^t(y) = \int_\Omega G_0^t(y)\mu(d\omega). \tag{1.2}$$

Our theorem asserts that the ratio $\sum_{k=0}^n F_k^s(x) / \sum_{k=0}^n G_k^t(y)$ converges almost everywhere uniformly in (x, y), which is however restricted to a set on which F^s, G^t behave with some moderation.

Theorem 1.1. *Let $s, t \varepsilon \bar{R}$ (extended real line). Let C and D be sets in $\bar{R}$ such that for some positive constants c, d*

$$C = \{x : F^s(x) \leqq c\}, \qquad D = \{y : G^t(y) \geqq d\}. \tag{1.3}$$

Let $B = C \times D$ and

$$\Delta_n = \sup_{(x,y)\varepsilon B} |(\textstyle\sum_{i=0}^{n-1} F_i^s(x) / \sum_{i=0}^{n-1} G_i^t(y)) - (F^s(x)/G^t(y))|. \tag{1.4}$$

Then for almost all $\omega \varepsilon \Omega$

$$\lim_{n\to\infty} \Delta_n = 0. \tag{1.5}$$

We note that Theorem 1 implies the Glivenko-Cantelli theorem (see [9], p. 335, [7], p. 20, Tucker [10]; also Fortet and Mourier [2]). Let μ be a probability measure and let $X_0 = Y_0$. Further set $s = t = -y = -\infty$ and $c = d = 1$. Then the denominator in the first ratio in (1.4) is simply n and Theorem 1.1 asserts the uniform convergence a.e. of the experimental distribution function $n^{-1} \sum_{i=0}^{n-1} F_i^{-\infty}(x)$ of a strictly stationary ergodic process (X_n), to the distribu-

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