ON THE EXACT DISTRIBUTIONS OF THE CRITERION W FOR TESTING SPHERICITY IN A p-VARIATE NORMAL DISTRIBUTION

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1. Introduction. Let the p-component vectors $x_1, x_2, x_3, \cdots, x_n$ form a sample from $N(\mu, \Sigma)$. The hypothesis H that $\Sigma = \sigma^2 I$, where σ^2 is not specified, can be put either in the form that all the roots of

$$|\Sigma - \phi I| = 0 \tag{1.1}$$

are equal, or that the arithmetic mean of the roots $\phi_1, \phi_2, \cdots, \phi_p$ is equal to the geometric mean, i.e.

$$\prod \phi_i^{1/p} / \{(\sum \phi_i)/p\} = |\Sigma|^{1/p} / \{(\operatorname{tr} \Sigma)/p\} = 1. \tag{1.2}$$

Since the squares of the lengths of principal axes of ellipsoids of constant density are proportional to the roots ϕ_i, which are now equal, the hypothesis implies that the ellipsoids are spheres.

If the covariance matrix A, for the sample, be given by

$$A = \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' = (a_{ij}) \tag{1.3}$$

the criterion W for testing sphericity in the p-variate normal distribution can be defined by

$$W = A / \{(\operatorname{tr} A)/p\}^p \tag{1.4}$$

which resembles (1.2). Thus the criterion W is a power of the ratio of the geometric mean and the arithmetic mean of the roots $\theta_1, \theta_2, \cdots, \theta_p$ of $|A - \theta I| = 0$.

Mauchly [9] defined a significance test for finding the ellipticity in a harmonic dial. In a subsequent paper [10] he modified his test to define a criterion for determining the sphericity of a normal p-variate distribution and also obtained its moments under the null hypothesis. Girshick [6] obtained the distribution of the ellipticity statistic under some special conditions.

Hickman [7] has given an example for obtaining the confidence regions for the dispersion matrix if it is taken to be proportional to any given matrix. Ihm [8] has discussed a number of such criteria in the case of multivariate normal distributions.

Anderson [1] has given a nice exposition of these different criteria satisfying different needs, the moments of such criteria and their distributions and the asymptotic expansions of the distributions. The hth moment of the sphericity

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