BOUNDED EXPECTED UTILITY

BY PETER C. FISHBURN

Research Analysis Corporation, McLean, Virginia

1. Introduction. The Blackwell-Girshick utility axioms [1], pp. 104–110, apply a preference-indifference relation $\precsim$ ("is not preferred to") to the set $\mathcal{P}_d$ of all discrete probability distributions defined on a set of consequences X. More precisely, with reference to a σ-algebra on X that contains $\{x\}$ for each $x \in X$, $\mathcal{P}_d$ is the set of all countably additive measures on the σ-algebra such that $P(A) = 1$ for some countable set A in the σ-algebra. The first purpose of this paper is to show that the Blackwell-Girshick utility theorem, which can be viewed as an extension of the standard von Neumann-Morgenstern result [3], can be obtained even on weakening their (B-G) denumerable "sure-thing" axiom. The second purpose is to show that versions of the new axiom, which is related to Savage's *P*7 [2], p. 77, can be used in deriving the expected-utility property for other sets of probability measures on X, including general σ-additive measures and finitely-additive measures. Bounded utilities result in all cases considered except for the case where all distributions are simple.

2. The von Neumann-Morgenstern theory. The von Neumann-Morgenstern expected-utility theory serves as the base of our discussion.

Let $\bar{\alpha} = (1 - \alpha)$ when $\alpha \in [0, 1]$. An *abstract convex set* is a set $\mathcal{P} = \{P, Q, R, \cdots\}$ and an operation $\alpha P + \bar{\alpha} Q$ associating an element of $\mathcal{P}$ with each fraction in $[0, 1]$ and each ordered pair of elements of $\mathcal{P}$, such that if $P, Q, R \in \mathcal{P}$ and $\alpha, \beta \in [0, 1]$ then

1. $1P + 0Q = P$,
2. $\alpha P + \bar{\alpha} Q = \bar{\alpha} Q + \alpha P$,
3. $\alpha(\beta P + \bar{\beta} Q) + \bar{\alpha} Q = \alpha\beta P + (1 - \alpha\beta) Q$.

With $\precsim$ a binary relation on $\mathcal{P}$, let $P \prec Q \Leftrightarrow [P \precsim Q$ and not $Q \precsim P]$, and $P \sim Q \Leftrightarrow [P \precsim Q$ and $Q \precsim P]$. $\precsim$ on $\mathcal{P}$ is a *weak order* if it is transitive and strongly connected ($P, Q \in \mathcal{P} \Rightarrow P \precsim Q$ or $Q \precsim P$).

The following axioms and theorem (proofs in [3], Appendix, and [2], Chapter 5) form the core of the theory. In all cases $P, Q, R \in \mathcal{P}$.

AXIOM 0. *$\mathcal{P}$ is an abstract convex set.*

AXIOM 1. *$\precsim$ on $\mathcal{P}$ is a weak order.*

AXIOM 2. $[P \sim (\prec) Q, \alpha \in (0, 1)] \Rightarrow \alpha P + \bar{\alpha} R \sim (\prec) \alpha Q + \bar{\alpha} R$.

AXIOM 3. $[P \prec Q, Q \prec R] \Rightarrow \alpha P + \bar{\alpha} R \prec Q$ *and* $Q \prec \beta P + \bar{\beta} R$ *for some* $\alpha, \beta \in (0, 1)$.

THEOREM 1. [*Axioms* 0, 1, 2, 3] $\Rightarrow$ *there is a real function u on $\mathcal{P}$ such that if $P, Q \in \mathcal{P}$ and $\alpha \in [0, 1]$ then*

$$u(P) \leqq u(Q) \Leftrightarrow P \precsim Q, \tag{1}$$

$$u(\alpha P + \bar{\alpha} Q) = \alpha u(P) + \bar{\alpha} u(Q). \tag{2}$$

Received 4 March 1965; revised 24 January 1966.