A SEQUENTIAL ANALOGUE OF THE BEHRENS-FISHER PROBLEM

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1. Introduction. In [1] and [4] a sequential procedure for the fixed-width interval estimation of the mean of a single population was investigated. We consider here an analogous procedure for estimating the difference of the means of two populations.

Let $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ be two independent sequences of rv's, the x's iid $N(\mu_1, \sigma_1^2)$ and the y's iid $N(\mu_2, \sigma_2^2)$. The four parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$ are assumed unknown. We want to find a confidence interval I of width $2d$ and with coverage probability $\geqq \alpha$ for the parameter $\Delta = \mu_1 - \mu_2$, where $0 < d < \infty$ and $0 < \alpha < 1$ are preassigned constants.

If σ_1, σ_2 were known we could proceed as follows. Take r observations on x and s observations on y, and let

$$\bar{x}_r = 1/r \sum_1^r x_i, \qquad \bar{y}_s = 1/s \sum_1^s y_j$$

be the respective sample means. If

$$I = [\bar{x}_r - \bar{y}_s - d, \bar{x}_r - \bar{y}_s + d] \tag{1}$$

is the interval of width $2d$ centered at $\bar{x}_r - \bar{y}_s$, then

$$P(\Delta \varepsilon I) = 2\Phi(d/(\sigma_1^2/r + \sigma_2^2/s)^{\frac{1}{2}}) - 1,$$

where Φ denotes the normal $(0, 1)$ df. Hence, defining the constants a and b by

$$2\Phi(a) - 1 = \alpha, \qquad b = (a/d)^2,$$

we have $P(\Delta \varepsilon I) \geqq \alpha$ providing that r, s satisfy the inequality

$$\sigma_1^2/r + \sigma_2^2/s \leqq 1/b. \tag{2}$$

Regarding r, s as continuous variables, the pair (r^*, s^*) which satisfies (2) and for which $n = r + s$ is a minimum is given by

$$r^* = b\sigma_1(\sigma_1 + \sigma_2), \qquad s^* = b\sigma_2(\sigma_1 + \sigma_2). \tag{3}$$

For this pair

$$r^*/s^* = \sigma_1/\sigma_2, \tag{4}$$

and the total sample size is

$$n^* = r^* + s^* = b(\sigma_1 + \sigma_2)^2. \tag{5}$$

We shall now give a sequential procedure when σ_1, σ_2 are unknown for determining r, s as random variables in such a way that (3) will hold approximately with

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