The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 266–269

THE ASYMPTOTIC ERROR OF ITERATIONS[1]

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0. Iterative methods appear in many mathematical investigations. The theorem concerning contraction mappings is a well known fact about iterations; it is useful for existence theorems and for estimates of error in numerical work. This paper investigates iterative processes in the presence of random errors; it is shown that these errors stabilize.

In Section one, a general theorem is formulated and proven for the case the basic space is the real numbers. In Section two, the theorem is generalized to Banach spaces. In Section three, connections with Markov processes and nonlinear integral equations are pointed out.

1. In the statement and proof of the following theorem, we use the notation $\|x\|$ instead of $|x|$ to anticipate the generalization to Banach spaces.

THEOREM 1. *Let*

(a) *$T(x)$ be a real function of a real variable satisfying*

$$\|T(x) - T(y)\| < K\,\|x - y\|, \quad \text{for all } x \text{ and } y, \text{ where} \quad K < 1; \tag{1}$$

(b) *$[\epsilon_i]_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables with $E(\|\epsilon_i\|) < \infty$;*

(c) *$[\delta_i]_{i=1}^{\infty}$ be a sequence of random variables with $\sum_{i=1}^{n} K^{n-1}\|\delta_i\|$ converging to zero in probability (or, equivalently, in distribution);*

(d) *the sequence $[D_n(X)]_{n=0}^{\infty}$ be defined by the following relations for each random variable X:*

$$\begin{aligned} D_0(X) &= X, \\ D_n(X) &= T[D_{n-1}(X)] + \epsilon_n + \delta_n \quad \text{for} \quad n \geqq 1. \end{aligned} \tag{2}$$

Under these conditions, the random variables $D_n(X)$ converge in distribution; the limiting distribution is determined by the function T and the common distribution of the random variables ϵ_i and thus does not depend on X and the random variables δ_i.

In the proof of the theorem, the following lemma is needed.

LEMMA. *Let $X_1, \cdots, X_n, \cdots$ be a sequence of random variables with $E(|X_i|) \leqq H < \infty$. Let K be a real number with $|K| < 1$. Then the series $\sum_{n=1}^{\infty} K^n X_n$ converges absolutely almost everywhere.*

PROOF. This follows directly from the inequalities:

$$E(\textstyle\sum_{n=1}^{N}|K^n X_n|) \leqq \sum_{n=1}^{N}|K|^n H \leqq |K|\cdot H/(1 - |K|) \quad \text{for every} \quad N.$$

PROOF OF THE THEOREM. The case with $\delta_n \neq 0$ can be reduced to the case $\delta_n \equiv 0$ as follows: Define $C_n(X)$ by

Received 25 March 1966; revised 19 July 1967.

[1] Supported by the U. S. Army Research Office (Durham) Contract No. DA-31-124-ARO(D)-87.