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DESIGNS FOR REGRESSION PROBLEMS WITH CORRELATED ERRORS; MANY PARAMETERS

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1. Introduction. Suppose one may observe a stochastic process $Y(\cdot)$ having the form

$$Y(t) = \sum_{j=1}^{J} \beta_j f_j(t) + X(t), \qquad t \varepsilon [0, 1]$$

where the β_j's are unknown parameters, the f_j's are known functions and $X(\cdot)$ is a stochastic process with mean function zero and known covariance kernel R. Under minor conditions on the regression functions f_j and the kernel R, one may for a suitably large finite observation set T, give expression to the best linear estimate (BLE) $\hat{\beta}$ of $\beta = (\beta_1, \cdots, \beta_J)'$ and to the covariance matrix of this estimate, say A_T^{-1}.

In a previous paper [2], we treated the following design set-up for the case $J = 1$ above: if $D_n = \{T \mid T = \{t_1, \cdots, t_n\}, 0 \leqq t_1 < \cdots < t_n \leqq 1\}$, an optimum design in D_n is a set T^* which minimizes A_T^{-1} over D_n (here A_T^{-1} is simply the variance of the BLE of β_1 based on the observation set T). In [2], we discussed the question of existence of optimum designs and, under certain restrictions, we produced sequences of designs $\{T_n^*\}$, $T_n^* \varepsilon D_n$, asymptotically optimum as $n \to \infty$. The necessity of pursuing such an asymptotic theory is discussed at some length in the introduction to [2].

In the present paper, we consider the cases $J > 1$. The fundamental difference here is, of course, that A_T^{-1} is a $J \times J$ matrix. Since one cannot expect the minimum of A_T^{-1} to exist over D_n (minimum in the sense of the ordering of non-negative definite matrices), our current problems arise from the attempt to minimize certain one-dimensional measures of the size of A_T^{-1}. The criteria treated below include, for example, the variance of $\theta'\hat{\beta}$ (viz. $\theta' A_T^{-1}\theta$) where θ is a fixed vector, the maximum variance of $\theta'\hat{\beta}$ with the maximum taken over a compact set $\mathfrak{M}$ of vectors, and the generalized variance det A_T^{-1}. We also consider certain regret criteria where regret is measured relative to what could be achieved through use of the observation set [0, 1].

Our basic assumptions are stated in Section 2 together with various formulations of the optimality of designs and the asymptotic optimality of sequences of designs. The necessary asymptotic results for this study are presented in Section 3 and, in the final section, they are applied to find asymptotically optimum sequences of designs for a variety of criteria. The asymptotically optimum sequences found are all, what we call below, regular. That is, the $(n + 1)$st de-

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