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NOTES

A CLARIFICATION CONCERNING CERTAIN EQUIVALENCE CLASSES OF GAUSSIAN PROCESSES ON AN INTERVAL[1]

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Two Gaussian processes will be called *equivalent* if the measures they induce on path space are mutually absolutely continuous. In [1], we considered the set of Gaussian processes obtained from stationary processes on the real line by restriction to a finite interval. A complete (although slightly awkward) description was given of the equivalence class, within this set, of the restriction of a process with rational spectral density. More recently, in [3], [4], [5], this theorem was generalized by Rozanov. However, the statements and proofs given were inexact, and there was one serious omission in the proposed necessary and sufficient conditions. It is the purpose of this note to give a correct and reasonably concise statement of Rozanov's more general theorem, together with an example to show the necessity of the condition which he omitted. An example to the same effect has also been obtained by B. Eisenberg. We refrain from including any proofs; the interested reader can presumably construct his own by use of [1], [3], [4], [5]. The details have been written out in [2].

Theorem. Let $d\mu(\lambda) = f(\lambda)\,d\lambda$, where f is a function on the real line, and, for a certain fixed n, and positive constant c,

$$c^{-1}|f(\lambda)| \leqq (1+\lambda^2)^{-n} \leqq c\,|f(\lambda)|, \qquad -\infty < \lambda < \infty.$$

Let $\rho(t) = \int_{-\infty}^{\infty} e^{it\lambda}\,d\mu(\lambda)$. Let ν be any other finite nonnegative measure on the real line, and let

$$\sigma(t) = \int_{-\infty}^{\infty} e^{it\lambda}\,d\nu(\lambda).$$

Then the following conditions are necessary and sufficient that the two Gaussian processes with mean zero and parameter sets $[-T, T]$, and whose covariances $E\{X_sX_t\}$ are respectively $\rho(s-t)$ and $\sigma(s-t)$, be equivalent.

(a) The function $\tau = \rho - \sigma$ has, on the open interval $(-2T, 2T)$, $2n-1$ derivatives. The $(2n-1)$st derivative is an absolutely continuous function. Thus, $(1 - d^2/dt^2)\tau$ is an a.e. defined function, call it Φ. Furthermore,

$$\int_{-T}^{T}\int_{-T}^{T} |\Phi(s-t)|^2\,ds\,dt < \infty.$$

(b) If F is an entire function of exponential type $\leqq T$, $\int |F|^2\,d\mu < \infty$, and $F = 0$ ν — a.e., then F is zero.

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