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THE ϵ-ENTROPY OF CERTAIN MEASURES ON [0, 1][1]

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1. Introduction. The epsilon-entropy of a probability distribution on a metric space was introduced in [4]. If C is a countable covering of the space by measurable sets we write $\|C\| = \max_{A \varepsilon C}$ (diameter (A)), $\#(C)$ = number of sets in C and

$$H(C) = \sum_{A \varepsilon C} P(A) \log (P(A))^{-1}.$$

Then the epsilon-entropy H_ϵ is given by

$$H_\epsilon = \inf_{\|C\| \leqq \epsilon} H(C).$$

In this paper we derive estimates of the asymptotic behavior of H_ϵ for certain singular measures on [0, 1]. The metric will be the usual length and we will write $|A|$ for the length of an interval A.

It will be convenient to use the notation $\phi(x) = x \log 1/x$. The function ϕ is convex and has the property that if $p_i \geqq 0$, $\sum_1^n p_i = 1$ then $\sum_1^n \phi(p_i) \leqq \log n$.

The theorems of this paper give asymptotic comparisons of H_ϵ with $\log \epsilon^{-1}$ which is approximately the ϵ-entropy of Lebesgue measure on [0, 1]. The asymptotic ratios are given in terms of various information theoretic quantities.

2. Measures related to N-adic expansions. Let N be a fixed integer, $N \geqq 2$ and let $(a_i, i = 1, 2, \cdots)$ be a stationary ergodic stochastic process taking the values $0, 1, \cdots, N - 1$. We assume that no fixed sequence (a_i^0), $i = 1, 2, \cdots$, has positive probability. Define $k_i(x)$ for irrational x in [0, 1] by

$$x = \sum_{i=1}^\infty k_i(x) N^i$$

where the sum on the right is the N-adic expansion of x. Write

$$I_n(l_1, \cdots, l_n) = [x \mid k_1(x) = l_1, \cdots, k_n(x) = l_n],$$
$$I_n(x) = I_n(k_1(x), \cdots, k_n(x)).$$

The probability measure P associated with the process induces a measure, which we also call P, on [0, 1] through the formula

$$P(I_n(l_1, \cdots, l_n)) = P(a_1 = l_1, \cdots, a_n = l_n).$$

According to the Shannon-Macmillan-Breiman theorem

$$\lim_{n\to\infty} n^{-1} \log P(I_n(x)) = -h(P) \text{ a.e. } (P)$$

where $h(P)$ is the entropy of the shift operator.

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