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HOW TO MINIMIZE OR MAXIMIZE THE PROBABILITIES OF EXTINCTION IN A GALTON-WATSON PROCESS AND IN SOME RELATED MULTIPLICATIVE POPULATION PROCESSES[1]

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1. Introduction. Let us consider a given population of N individuals who form, say, the 0th generation, who produce during their lifetime new individuals who form the 1st generation, who in turn produce during their lifetime new individuals who form the 2nd generation, who in turn produce during their lifetime new individuals who form the 3rd generation, etc. Let q_{ij} be the probability that an individual in the jth generation ($j = 0, 1, 2, \cdots$) will produce during his lifetime i new individuals ($i = 0, 1, 2, \cdots$) in the $(j + 1)$th generation. We assume that within each generation, given the past, individuals reproduce independently of one another, and that $\sum_{i=0}^{\infty} q_{ij} = 1$ for $j = 0, 1, 2, \cdots$. Let μ_j denote the mean number of new individuals produced during his lifetime by an individual in the jth generation; i.e., $\mu_j = \sum_{i=0}^{\infty} iq_{ij}$, for $j = 0, 1, 2, \cdots$. We call μ_j the Malthusian rate for the jth generation (see, e.g., Karlin (1966), p. 364). Let $\mathbf{m} = \{m_0, m_1, m_2, \cdots\}$ and $\mathbf{M} = \{M_0, M_1, M_2, \cdots\}$ denote two sequences of numbers which are such that

$$m_j \leqq \mu_j \leqq M_j, \quad \text{for} \quad j = 0, 1, 2, \cdots. \tag{1}$$

For any given $\mathbf{m}$ and $\mathbf{M}$, we shall consider all possible values of q_{ij} which are such that Condition (1) is satisfied, and we shall answer the following four questions herein: (A) How can we minimize the probability that the jth generation ($j = 0, 1, 2, \cdots$) will become extinct? (B) How can we maximize the probability that the jth generation ($j = 0, 1, 2, \cdots$) will become extinct? (C) How can we minimize the probability of eventual extinction? (D) How can we maximize the probability of eventual extinction?

In a recent article, Freedman and Purves (1967) answered question (A) for the special case where it is assumed that

$$q_{1j} = 0, \quad \text{for} \quad j = 0, 1, 2, \cdots, \tag{2}$$

and that Condition (1) is satisfied with $m_j = 0$ and $M_j = M < 2$ for $j = 0, 1, 2, \cdots$. The answer to question (A) under the special restriction (2) differs from the corresponding answer obtained herein when (2) is not assumed. In

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