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ON INVARIANCE AND ALMOST INVARIANCE

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1. Introduction. The requirement that almost-invariant test statistics should be equivalent to invariant test statistics plays a central role in the theory of invariant and unbiased tests [5], sufficiency and invariance [4], etc. A classical condition for this equivalence to hold, due to Stein, may be found in [5], p. 225.

More recently, Bell [1] has given an approach which yields the desired result in nonparametric situations. The purpose of this note is primarily to show that the latter approach applies in most parametric cases as well. In addition, we give a decision-theoretic version of Stein's result.

2. The result. Let $(\mathfrak{X}, \mathcal{B})$ be the measurable (sample) space of the random variable X and $\mathcal{P}$, a family of distributions for X. We suppose $\mathcal{P}$ is generated by G, a group of bimeasurable transformations of $\mathfrak{X}$ to itself, i.e., $\mathcal{P} = \{Pg^{-1} : g \varepsilon G\}$ for any P in $\mathcal{P}$. We refer the reader to [5] for the definitions of terms used from this point on.

Let I be a measurable maximally invariant statistic inducing the invariant σ-field $\mathfrak{I} \subset \mathcal{B}$ and S, another measurable statistic with the induced σ-field $\mathcal{S} \subset \mathcal{B}$, so that the correspondence $X \leftrightarrow (I, S)$ is 1–1 bimeasurable. We suppose also that G acting on X induces a group of transformations, G_S, acting on S. That is, if $X \leftrightarrow (I, S)$, $gX \leftrightarrow (I, g_S S)$. For conditions under which this structure is present, see [3].

(1) THEOREM. *If S is sufficient and boundedly complete, then any $\mathcal{P}$-almost-invariant test function is $\mathcal{P}$-equivalent to an invariant one.*

The proof is preceded by two lemmas. In the sequel, ϕ will denote the test function $\phi(X)$. (ϕ is also called a critical function.) We note that if ϕ is almost-invariant, its distribution is independent of $P \varepsilon \mathcal{P}$. Hence we shall refer to the $\mathcal{P}$-distribution or expectation of almost invariant statistics. Similarly, we may refer to the conditional $\mathcal{P}$-expectation given the sufficient σ-field $\mathcal{S}$ and $E_{\mathcal{P}}(\phi \mid \mathcal{S})$ will denote an element of the $\mathcal{P}$-equivalence class containing $\{E_P(\phi \mid \mathcal{S}) : P \varepsilon \mathcal{P}\}$.

(2) LEMMA. *If ϕ is almost-invariant, $E_P(\phi \mid \mathfrak{I})$ is independent of* of $P \varepsilon \mathcal{P}$.

REMARK. The lemma is actually a special case of the more general fact (which we prove): if $\mathfrak{I}^*$ is the almost invariant σ-field and $\mathcal{C} \subset \mathfrak{I}^*$ is a σ-field, then $E_{\mathcal{P}}(\phi \mid \mathcal{C})$ is meaningful.

PROOF. Choose $P, Q \varepsilon \mathcal{P}$ and let $P = Qg^{-1}$. Then

$$E_P(\phi \mid \mathcal{C}) = E_Q(\phi g \mid g^{-1}\mathcal{C})g^{-1} = E_Q(\phi \mid \mathcal{C})g^{-1} = E_Q(\phi \mid \mathcal{C}) \ [Q].$$

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