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ALMOST SURE CONVERGENCE OF QUADRATIC FORMS IN INDEPENDENT RANDOM VARIABLES[1]

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In [3], we began a study of convergence of quadratic forms in independent random variables. Simultaneously, Fau Dyk Tin and G. E. Silov [2] initiated their study of this problem but restricted to the case of quadratic mean convergence and normal variables. Our aim in this paper is to consider carefully the problem of almost sure convergence (convergence with probability one). Several of our results will generalize well known theorems for series of independent random variables.

We shall assume throughout that $X_1, X_2, \cdots$ is a sequence of independent real random variables with $E(X_k) = 0$ and $E(X_k^2) = 1$, $k = 1, 2, \cdots$. Note that we do not assume that the X_k's are identically distributed or place conditions on the higher moments. Let (a_{jk}), $j, k = 1, 2, \cdots$, be a real (not necessarily symmetric) matrix and let

$$S_n = \sum_{j,k=1}^{n} a_{jk} X_j X_k .$$

At various times, we shall place special restrictions on (a_{jk}). We say that (a_{jk}) is *Hilbert-Schmidt* if $\sum_{j,k=1}^{\infty} a_{jk}^2 < \infty$, that it is *nuclear* if $a_{jk} = \sum_{i=1}^{\infty} b_{ji} c_{ik}$ where (b_{ji}) and (c_{ik}) are Hilbert-Schmidt, and that it is *positive semi-definite* if it is symmetric and $\sum_{j,k=1}^{n} a_{jk} u_j u_k \geqq 0$ for all choices of n and $u_1, \cdots, u_n$. We observe that the class of Hilbert-Schmidt matrices contains all nuclear matrices (use the Schwarz inequality) and all those positive semi-definite matrices with finite trace (use the inequality $\sum_{j,k=1}^{n} a_{jk}^2 \leqq (\sum_{k=1}^{n} a_{kk})^2$).

THEOREM 1. *If (a_{jk}) is Hilbert-Schmidt and $\sum_{k=1}^{n} |a_{kk}| < \infty$, then S_n converges almost surely.*

REMARK. This theorem is the best possible in the following sense. For any Hilbert-Schmidt matrix (a_{jk}) with $\sum_{k=1}^{\infty} |a_{kk}| = \infty$, there is a sequence $X_1, X_2, \cdots$ of independent random variables with $E(X_k) = 0$ and $E(X_k^2) = 1$ for which S_n diverges almost surely. We omit this fairly simple construction.

PROOF. Let K_n, L_n, and M_n be defined in the obvious manner by

$$\begin{aligned} S_n &= \sum_{j=1}^{n} X_j \sum_{k=1}^{j-1} a_{jk} X_k + \sum_{k=1}^{n} X_k \sum_{j=1}^{k-1} a_{jk} X_j + \sum_{k=1}^{n} a_{kk} X_k^2 \\ &= K_n + L_n + M_n . \end{aligned}$$

Now K_n is a martingale and since

$$[E(|K_n|)]^2 \leqq E(K_n^2) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} a_{jk}^2 \leqq \sum_{j,k=1}^{\infty} a_{jk}^2 ,$$

it follows that K_n converges almost surely (martingale convergence theorem).

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