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CONVERGENCE RATES FOR THE LAW OF THE ITERATED LOGARITHM

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Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables and S_n denote the partial sums $\sum_{k=1}^{n} X_k$. For such sequences of random variables, with mean zero and variance one, the Law of the Iterated Logarithm states that $P[\limsup S_n(2n \lg \lg n)^{-\frac{1}{2}} = 1] = 1$. In this paper probabilities where this law is applicable are considered and appropriate convergence rates are determined.

According to Levy's terminology, for a given sequence of independent random variables $\{X_n\}$ a monotonic sequence $\{\varphi_n\}$ is said to be in the lower class $\mathfrak{L}$ if $P[S_n > n^{\frac{1}{2}}\varphi_n$ infinitely often$] = 1$. Otherwise the above probability is zero and the sequence is said to be in the upper class $\mathfrak{U}$. In 1946 Feller [6] characterized the upper and lower classes for independent identically distributed random variables with $EX = 0$, $EX^2 = 1$, and $EX^2 \lg \lg |X| < \infty$. Namely, $\{\varphi_n\}$ is in the upper (lower) class if the series $\sum \varphi_n e^{-\varphi_n^2/2} n^{-1}$ converges (diverges).

In this paper, the initial results are directed toward obtaining a convergence rate for $P[\sup_{k \geqq n} |S_k[(2 + \epsilon)k \lg \lg k]^{-\frac{1}{2}}| > 1]$ for independent identically distributed random variables satisfying the above moment conditions. In Theorem 3 it is shown that under a somewhat stronger moment condition Feller's criterion $(\sum \varphi_n e^{-\varphi_n^2/2} n^{-1} < \infty)$ is equivalent to the convergence of $\sum \varphi_n^2 n^{-1} P[|S_n| > n^{\frac{1}{2}}\varphi_n]$. Finally, random variables with $EX = 0$ and $EX^2 = 1$ are considered and it is shown that a weaker criterion than Feller's is sufficient to guarantee a convergence rate for $P[|S_n| > n^{\frac{1}{2}}\varphi_n]$. Thus there are monotonic sequences $\{\varphi_n\}$ such that $P[S_n > n^{\frac{1}{2}}\varphi_n$ infinitely often$] = 1$ and yet the series $\sum n^{-1} P[|S_n| > n^{\frac{1}{2}}\varphi_n]$ converges. To obtain some of the preceding, extensive use has been made of results and techniques developed in [8] where Friedman, Katz, and Koopmans applied the convergence rate concept to the Central Limit Theorem.

In this paper $\{X_n\}$ denotes sequences of independent identically distributed random variables with common distribution function F; $\Phi(x)$ represents the standard normal distribution function, and $[x]$ will stand for the largest integer less than or equal to x. Also

$$\begin{aligned} \lg x &= \log_e x, \qquad & x > 1 \\ &= 0 & \text{otherwise.} \end{aligned}$$

RESULTS. Initially, it is easily observed that in [8] the following somewhat stronger result is actually proven.

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