The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2094–2097

NOTES

ON A THEOREM OF SKOROHOD[1]

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1. Skorohod in [2], p. 180, found for each L_2-martingale $X_1, X_2, \cdots$ of mean 0 with independent increments, a sequence of stopping times $\tau_1 \leqq \tau_2 \leqq \cdots$ for standard Brownian motion $B(t)$ such that (i): $B(\tau_1), B(\tau_2), \cdots$ has the same joint distribution as the martingale and (ii): each τ_j has a finite expectation. (And David Freedman and Strassen [3], p. 318, noted that the assumption of independent increments may be dropped.) The stopping times Skorohod found depend upon a random variable independent of the Brownian motion $B(t)$. The point of this note is to exhibit equally effective stopping times τ_j whose moment of stopping, $\tau_j(\omega)$, depends only on the path ω, and not on a random variable independent of the Brownian motion. The construction incidentally realizes the martingale $X_1, X_2, \cdots$ inside Brownian motion in a natural way even if the X_j do not have finite second moments, and indeed sometimes even if they have no first moment. What is essential is that $E[X_{n+1} - X_n \mid X_1, \cdots, X_n]$ be 0, in which event, the process is *fair*; but it is not essential that the increment $X_{n+1} - X_n$ itself have a mean. Moreover, as will also be evident, the *same* stopping times τ_j embed the discrete-time martingale $X_1, X_2, \cdots$ inside any continuous-time martingale $M(t)$ that resembles Brownian motion in having continuous, unbounded paths with $M(0) \equiv 0$.

2. Let $\mathcal{C}$ be the set of all continuous, real-valued functions ω defined for $0 \leqq t < \infty$ which are unbounded from above and from below. Of course, if $\mathcal{C}$ is endowed with its natural σ-field, on which a countably-additive probability is given, then $\omega(t)$, or, more precisely, the set of evaluation maps $\omega \rightarrow \omega(t)$, becomes a stochastic process. All continuous processes in this note are to be understood to be of this form.

A *lottery* is a probability measure on the real line.

The program is to define for every lottery μ with a finite expectation $E(\mu)$, and every $\omega \in \mathcal{C}$, a nonnegative real number $\tau(\mu, \omega) = \tau(\mu)(\omega)$, so that $\tau(\mu)$ is a stopping time such that, for every martingale $\omega(t)$ with unbounded, continuous paths, and with $\omega(0) \equiv E(\mu)$, the map $\omega \rightarrow \omega(\tau(\mu, \omega))$ has μ for its distribution.

To define $\tau(\mu)$, it is convenient to introduce μ^+ and μ^- for the conditional

Received 13 February 1968.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 1312-67.