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SEQUENTIAL COMPOUND ESTIMATION[1]

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1. Introduction. We consider a sequence of statistical decision problems having the same generic structure with this structure being possessed by what is called the component problem. In the component problem there is a family of probability measures $\{P_\theta \mid \theta \varepsilon \Omega\}$ over a σ-field $\mathcal{B}$ of subsets of $\mathfrak{X}$, an action space $\mathcal{A}$, and a loss function $L \geqq 0$ defined on $\Omega \times \mathcal{A}$. With $\mathcal{C}$ a σ-field of subsets of $\mathcal{A}$, a (randomized) decision function φ has domain $\mathfrak{X} \times \mathcal{C}$ and is such that $\varphi(x, \cdot)$ is a probability measure on $\mathcal{C}$ for each fixed $x \varepsilon \mathfrak{X}$ and $\varphi(\cdot, C)$ is $\mathcal{B}$ measurable for each fixed $C \varepsilon \mathcal{C}$. The decision procedure φ results in an expected loss (risk)

$$R(\theta, \varphi) = \int \int L(\theta, A)\varphi(x, dA)P_\theta\,(dx). \tag{1.1}$$

In treating the sequence of component problems it is convenient to introduce the notation $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots)$ and $\boldsymbol{\theta}_i = (\theta_1, \cdots, \theta_i)$; also, we assume that $\mathbf{X}_i \sim P_{\theta_1} \times \cdots \times P_{\theta_i} = \mathbf{P}_i$ for all $i \geqq 1$. The action taken at the ith stage (i.e., in the ith repetition of the component problem) is allowed to depend on $\mathbf{X}_i$. Formally, a sequential compound procedure $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$ is such that for each i, φ_i is the means by which the ith action is taken, φ_i is defined on $\mathfrak{X}^i \times \mathcal{C}$ with $\varphi_i(\cdot, C)\mathcal{B}^i$ measurable for each C, and $\varphi_i(\mathbf{x}_i, \cdot)$ is a probability measure on $\mathcal{C}$ for each $\mathbf{x}_i$. The average risk up to stage n is

$$R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1}\textstyle\sum_{i=1}^n \int \int L(\theta_i, A)\varphi_i(\mathbf{x}_i, dA)\mathbf{P}_i\,(d\mathbf{x}_i). \tag{1.2}$$

In keeping with terminology that is becoming standard, we say a sequential compound procedure $\boldsymbol{\varphi}$ is simple if $\varphi_i(\cdot, C)$ is x_i measurable for each C. If in addition all the φ_i are identical, say $\varphi_i = \varphi$, we say $\boldsymbol{\varphi}$ is simple symmetric with kernel φ. Simple symmetric procedures are traditional in case Ω is not a singleton set. For every simple symmetric procedure $\boldsymbol{\varphi}$ and all $\boldsymbol{\theta}$,

$$R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1}\textstyle\sum_{i=1}^n R(\theta_i, \varphi) \geqq R(G_n) \tag{1.3}$$

where G_n is the empirical distribution of $\theta_1, \cdots, \theta_n$ and $R(\cdot)$ is the Bayes envelope for the component problem. We also note that for any simple procedure $\boldsymbol{\varphi}$,

$$\begin{aligned}\sup_{\boldsymbol{\theta}}\{R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G_n)\} &\geqq \sup_\theta \{n^{-1}\textstyle\sum_{i=1}^n R(\theta, \varphi_i) - \inf_A L(\theta, A)\} \\ &\geqq \inf_\varphi \sup_\theta \{R(\theta, \varphi) - \inf_A L(\theta, A)\}.\end{aligned} \tag{1.4}$$

(Samuel (1965b) gives a necessary condition for the left hand side of (1.4) to be zero.) The right hand side of (1.4) is zero only when the component problem is trivial; otherwise, it is some positive number, say ϵ. Therefore, with a modified

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