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EXISTENCE OF AN INVARIANT MEASURE AND AN ORNSTEIN'S ERGODIC THEOREM[1]

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0. Introduction. T being a Markovian positive operator acting on $L_1(\lambda)$, and prolongated to the space $M^+(\lambda)$ of all λ-equivalence classes of positive functions, we are looking for finite $f \varepsilon M^+(\lambda)$ such that $Tf = f$. By using a very skillful and deep construction due to Ornstein (cf. [8], Part III), we give an existence theorem of such a T-invariant f, T belonging to a suitable class of conservative Markovian operators (Theorem 1), and we make clear a general setting in which an ergodic theorem proved by D. Ornstein for random walks ([8]) can be stated. Namely, for any bounded function h with a suitably "bounded" support and verifying $\int h\,d\lambda = 0$, the function $\sup_n |\sum_{i=1}^n T^i h|$ is bounded by pf, p constant.

Beside this situation, which for our convenience we call "the abstract case", we are looking in Section 3 at the problem of finding a σ-finite ν such that $\nu = \nu \underline{P}$ (cf. Definition in 1.7 through 1.9), where $\underline{P}$ is a Markov kernel, and state the previous ergodic theorem in this situation which we call "concrete case." We prove in 3.2 that our hypotheses in the "concrete case" are essentially equivalent to those of Harris' theorem on invariant measures. (Cf. [2] and our Theorem 4.) We give thus an alternate proof of the Harris theorem. Moreover, by introducing natural topological hypothesis, when E is locally compact, we can state that the invariant measure is a regular Borel measure, and we are allowed to use the word "bounded" above in the usual topological sense (i.e. with compact closure. Cf. Theorem 5).

The Part 4 is devoted to ratio limit theorems for Markov kernels, strengthening N. C. Jain's result insofar as we prove that, in some cases, "almost everywhere" can be replaced by "everywhere" in Jain's statements. In fact it is proved in [5] that, if $\underline{P}$ is any Harris Markovian kernel, and if λ is a $\underline{P}$-invariant measure, for any A and B measurable with $\lambda(B) < +\infty$,

$$\lim_{n\to\infty} \sum_{k=0}^n \underline{P}^k(x, A)/\sum_{k=0}^n \underline{P}^k(y, B) = \lambda(A)/\lambda(B) \qquad \text{for every } x \text{ and } y$$

outside a λ-null set (depending on A and B). We prove in fact (Theorem 6 and 7) that, if A and B are "bounded sets" (Cf. Definition 4.1) the above limit holds for every x and y.

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