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ON ERLANG'S FORMULA[1]

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1. Introduction. The following mathematical model of telephone traffic has some importance in designing telephone exchanges.

In the time interval $(0, \infty)$ calls are arriving at a telephone exchange in accordance with a Poisson process of density λ, that is, if $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ denote the arrival times, then $\tau_n - \tau_{n-1} (n = 1, 2, \cdots; \tau_0 = 0)$ are mutually independent random variables having a common distribution function

$$(1) \qquad F(x) = 1 - e^{-\lambda x} \quad \text{if} \quad x \geqq 0, \qquad = 0 \quad \text{if} \quad x < 0.$$

There are m available lines. If an arriving call finds a free line, then a connection is realized without delay. If every line is busy when a call arrives, the call is lost. The holding times are mutually independent, positive random variables having a common distribution function $H(x)$, and a finite expectation

$$(2) \qquad \alpha = \int_0^\infty x \, dH(x).$$

The holding times are also independent of the arrival times and the initial state. The initial state is given by the number of busy lines at time $t = 0$ and by the remaining lengths of the holding times in progress at time $t = 0$.

If we choose the initial distribution in such a way that the process becomes stationary, then the probability that at time t the number of busy lines in k is given by Erlang's formula,

$$(3) \qquad P_k = [(\lambda\alpha)^k/k!][\textstyle\sum_{j=0}^m (\lambda\alpha)^j/j!]^{-1}$$

for $k = 0, 1, \cdots, m$ and all $t \geqq 0$.

This formula has an interesting history. In 1917 A. K. Erlang [1] deduced formula (3) for the case when the holding times are constant α. While Erlang's result is correct, his proof is not complete. He has made use of a property of the process which is far from evident. Erlang noted also that if the holding times have an exponential distribution, that is,

$$(4) \qquad H(x) = 1 - e^{-x/\alpha} \quad \text{for} \quad x \geqq 0, \qquad = 0 \quad \text{for} \quad x < 0,$$

then (3) is valid. If $H(x)$ is an exponential distribution function, then Erlang's proof is acceptable.

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