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ON MEASURABLE, NONLEAVABLE GAMBLING HOUSES WITH A GOAL[1]

By William D. Sudderth

University of California, Berkeley

1. Introduction. A gambler's problem, as formulated by Dubins and Savage in [1], consists of a set F of fortunes, a bounded utility function u on F to the real numbers, and, for each f in F, a set $\Gamma(f)$ of gambles (finitely additive probability measures defined on all subsets of F). A strategy σ available in the gambling house Γ at the fortune f is a sequence $\sigma_0, \sigma_1, \cdots$ where $\sigma_0 \varepsilon \Gamma(f)$ and, for $n \geqq 1$, σ_n is a gamble-valued function defined on $F \times F \times \cdots \times F$($n$-factors) such that $\sigma_n(f_1, \cdots, f_n) \varepsilon \Gamma(f_n)$ for every partial history $(f_1, \cdots, f_n)$. The strategy σ may be regarded as a probability measure defined on the finitary subsets of the infinite product $H = F \times F \times \cdots$ and $\sigma_n(f_1, \cdots, f_n)$, as the conditional σ-distribution of f_{n+1} given $(f_1, \cdots, f_n)$ (Section 2.8 of [1]). A gambler with fortune f chooses an available strategy σ and a stop rule t and gets a return $u(\sigma, t)$, the expected value of $u(f_t)$ under σ. By $U(f)$ is denoted the maximum of $u(f)$ and the sup $u(\sigma, t)$ taken over all available σ and stop rules t. Strauch has shown in [4] that if a gambling problem is assumed to have a certain natural Borel measurability structure, then U is measurable with respect to the completion of any Borel measure on the Borel subsets of F and there exist good Borel measurable strategies (See also [5] and [6]).

If a gambler using the strategy σ is not allowed to terminate play, he receives $u(\sigma) = \lim \sup_{t\to\infty} u(\sigma, t)$. $V(f)$ is the sup $u(\sigma)$ taken over all strategies σ available at f. If Γ is leavable, that is, if the one-point gamble $\delta(f)$ is in $\Gamma(f)$ for all f, then $V = U$ ([1], Corollary 3.3.2, p. 42). If Γ has the Borel measurability structure assumed by Strauch and is not leavable, it is not known whether V is absolutely measurable or if good measurable strategies exist.

In this note, I treat the special case in which the utility function u is the indicator of a single fortune g called the goal. It is seen that $u(\sigma)$ may be interpreted as the "σ-probability of visiting g infinitely often" and the questions above are settled affirmatively.

Unless otherwise indicated, the terminology and notation of this note are intended to have the same meaning as in [1].

2. Measurable strategies and probability measures on H. Assume F is a Borel subset of a complete separable metric space and let $\mathcal{B}$ denote the Borel subsets of F. Let Γ be a gambling house defined on F such that every gamble available in Γ is countably additive when restricted to $\mathcal{B}$. (Let P be the set of countably

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