The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 532–539

QUASI-STATIONARY BEHAVIOUR OF A LEFT-CONTINUOUS RANDOM WALK

By D. J. Daley[1]

University of Washington

1. Introduction. Consider a random walk $\{S_n\}$ $(n = 0, 1, \cdots)$ on the integers $\{\cdots, -1, 0, 1, \cdots\}$ for which $S_0 = 1$ and

$$\text{pr}\,\{S_{n+1} = S_n + k \mid S_n\} = p_k \qquad (\text{all } n, k) \tag{1.1}$$

such that

$$p_{-1} > 0, \qquad p_k = 0\ (k = -2, -3, \cdots), \qquad \textstyle\sum_{k=-1}^{\infty} p_k = 1. \tag{1.2}$$

The main object of this note is to study the limits as $n \to \infty$ of

$$a_j^n = \text{pr}\,\{S_n = j \mid \min(S_1, \cdots, S_n) > 0, S_0 = 1\} \tag{1.3}$$

when

$$0 < m = 1 + \textstyle\sum_{k=-1}^{\infty} kp_k < 1, \tag{1.4}$$

the limits being zero when $m \geqq 1$. In other words, if after a long time the process has not yet visited the set $\{\cdots, -1, 0\}$ what (if any) is its asymptotic behaviour? An extensive discussion of such questions in the context of Markov chains on a countable state space is given in papers by Seneta and others, the most refined results being given in Seneta and Vere-Jones (1966). This note may be regarded as an illustration of their work in the case of a moderately simple Markov chain, or as an addendum to what is already known on left-continuous simple random walks. To simplify our discussion, we assume that

$$\{S_n\} \quad \text{is aperiodic, i.e.,} \quad \gcd\,\{j : p_{j-1} > 0\} = 1. \tag{1.5}$$

In the trivial case that $p_{-1} + p_0 = 1$ and $p_{-1} < 1$, $a_j^n = 1$ if $j = 1$ and $= 0$ otherwise, so to eliminate this exception we assume further that

$$p_{-1} + p_0 < 1. \tag{1.6}$$

With this notation and

$$f(s) = \textstyle\sum_{k=-1}^{\infty} p_k s^{k+1} \qquad (|s| \leqq 1), \tag{1.7}$$

we shall prove

Theorem 1. *For a left-continuous aperiodic random walk $\{S_n\}$ with mean step-length $m - 1 < 0$,*

$$\lim_{n\to\infty} a_j^n = \lim_{n\to\infty} \text{pr}\,\{S_n = j \mid S_r > 0\ (r = 1, \cdots, n), S_0 = 1\}$$

Received 23 January 1968; revised 26 August 1968.

[1] Work done on leave from Business Operations Research Ltd Research Fellowship, Selwyn College, Cambridge, England.