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AN APPLICATION OF THE SOBOLEV IMBEDDING THEOREMS TO CRITERIA FOR THE CONTINUITY OF PROCESSES WITH A VECTOR PARAMETER[1]

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1. Introduction. Let $f(x)$ be a real valued random process with parameter x, and whose parameter set $\bar{R}$ is a set in E^n (Euclidean n-space). Following the usual terminology, a version of $f(x)$ is any process $\tilde{f}(x)$, defined for $x \varepsilon \bar{R}$, which satisfies $P\{f(x) = \tilde{f}(x)\} = 1$ for $x \varepsilon \bar{R}$. It is sometimes useful to know that there exists a version of $f(x)$ which is wp 1. (with probability one) continuous, Holder continuous, or perhaps differentiable in some component on $\bar{R}$. In the sequel, we give some criteria for these properties. The criteria are somewhat analogous to the criteria, depending on integrability of certain 'weak' derivatives, for the continuity of a sure function (see Smirnov [4], Sec. 114–118). The work was motivated by some problems in stochastic control theory of which one is very briefly discussed in the example of Section 5.

The results involve notions of separability and measurability for vector parameter processes. The applicability of Doobs arguments concerning the existence of separable or measurable versions was noted in Doob [1] (his remark preceding Lemma 2.1, Chapter 2). Although the exact form of the required results does not appear to have been stated, the proof of our Theorem 1 is almost identical in form to that of Neveu [3], p. 91–92.

2. Separability and measurability. N, N_i or $N(y)$ denote null ω-sets, where ω is the generic variable of the sample space. Let R be a bounded open set with closure $\bar{R}$. Let $f(x)$ be a family of real random variables whose parameter x is defined on some domain $\bar{R} \subset E^n$ (Euclidean n-space). Let A be an arbitrary open set in E^n with closure $\bar{A}$ and write $f(A) = \bigcup_{x\varepsilon A} f(x)$. If there is an N and a dense (in $\bar{R}$) denumerable set $\mathfrak{I} \subset \bar{R}$ so that, for each $x \varepsilon \bar{R}$ and each $\omega \not\varepsilon N$,

$$f(x) \varepsilon \bigcap_{x\varepsilon A} \overline{f(\mathfrak{I} \cap A)}$$

where the intersection is taken over all open A containing x, then the process $f(\cdot)$ is said to be *separable*, with *separability set* $\mathfrak{I}$.

This (vector parameter) definition is a natural analog of the scalar parameter definition of separability of Neveu [3]. If $f(\cdot)$ is separable then (the analog of the definition of separability of Doob [1]) there is some null set N so that for any compact real interval Γ, and any open set A, the ω sets

$$\{f(x) \varepsilon \Gamma, x \varepsilon A \cap \bar{R}\}, \qquad \{f(x) \varepsilon \Gamma, x \varepsilon A \cap \bar{R} \cap \mathfrak{I}\}$$

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