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MAXIMUM LIKELIHOOD ESTIMATION OF MULTIVARIATE COVARIANCE COMPONENTS FOR THE BALANCED ONE-WAY LAYOUT

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1. Introduction. Unbiased estimators of variance and covariance components for the balanced one-way layout have been extensively investigated in the literature. Unfortunately, they possess the unpleasant property of taking on inadmissible values such as negative variances and, more generally, non-positive-semidefinite covariance matrices. This in turn can lead to correlation coefficients that are imaginary or greater than one.

In the univariate case, the maximum likelihood (ml) estimators, which are free from these drawbacks, have been derived by Herbach [3] and shown in [5] to have uniformly, and in many cases considerably, smaller mean square errors than the unbiased estimators. Hence it is of interest to consider ml estimation in the multivariate case. Searle [7] computed the information matrix for the bivariate case, but did not derive explicit expressions for the estimators.

In this paper, we define (in Section 2) and derive (in Section 3) the maximum likelihood estimators for the geneal P-variate case. In Section 4 the methods of computation are described, and in Section 5 explicit formulae are given for the bivariate case.

2. Model, notation, and extended definition of ml estimators. Denote the P-variate observation row vectors by $\mathbf{x}_{jk}$. The variance component model corresponding to the balanced one-way layout is

$$\mathbf{x}_{jk} = \boldsymbol{\mu} + \mathbf{b}_j + \mathbf{w}_{jk} \qquad (j = 1, 2, \cdots, J; k = 1, 2, \cdots, K), \tag{2.1}$$

where $\boldsymbol{\mu}$ is a fixed mean vector, and the $J(K+1)$ random multinormal vectors $\mathbf{b}_j : N(\mathbf{0}, \mathbf{\Sigma}_b)$ and $\mathbf{w}_{jk} : N(\mathbf{0}, \mathbf{\Sigma}_w)$ are independent. The within-groups covariance matrix $\mathbf{\Sigma}_w$ is assumed to be positive definite (pd), but the between-groups covariance matrix $\mathbf{\Sigma}_b$ may be positive semidefinite (psd). Denote $\mathbf{x}_{j\cdot} = \Sigma_k \mathbf{x}_{jk}/K$ and $\mathbf{\Gamma} = \mathbf{\Sigma}_w + K\mathbf{\Sigma}_b$. Reduction of the sample space by sufficiency, using the factorization theorem, yields the complete sufficient statistic $(\mathbf{x}_{\cdot\cdot}, \mathbf{S}_b, \mathbf{S}_w)$ defined by

$$\begin{aligned} \mathbf{x}_{\cdot\cdot} &= \textstyle\sum_j \sum_k \mathbf{x}_{jk}/(JK), \\ \mathbf{S}_b &= K \textstyle\sum_j (\mathbf{x}_{j\cdot} - \mathbf{x}_{\cdot\cdot})'(\mathbf{x}_{j\cdot} - \mathbf{x}_{\cdot\cdot}) \qquad \text{and} \\ \mathbf{S}_w &= \textstyle\sum_j \sum_k (\mathbf{x}_{jk} - \mathbf{x}_{j\cdot})'(\mathbf{x}_{jk} - \mathbf{x}_{j\cdot}). \end{aligned}$$

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