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UNIFORM CONVERGENCE OF FAMILIES OF MARTINGALES[1]

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It is known [3] that under bounded conditions on the expected values of a martingale sequence the martingale will converge almost surely to a random variable and if the rth powers of the absolute values of the random variables in the martingale are uniformly integrable then the martingale will converge in L_r to a random variable with finite rth absolute moment. In this note we consider the case of a family of martingales each adapted to the same increasing family of σ-fields and give a condition on the family which will assure under bounded conditions on the martingales that the convergence given by the martingale convergence theorem is uniform in the family. We obtain uniform L_1 convergence for arbitrary families and uniform a.s. convergence for countable families. The a.s. convergence was proven for a slightly different case in [4] and the L_1-convergence is obtained from a suggestion made by the referee of that paper.

Throughout we will be working in a fixed probability space $(\Omega, \mathcal{B}, P)$ and all σ-fields will be sub σ-fields of $\mathcal{B}$. If A and B are sets $A \triangle B$ will denote the symmetric difference of A and B, i.e., $A \triangle B = (A - B) \cup (B - A)$. The expected value of a random variable will be denoted by E, the conditional expectation given a σ-field $\mathcal{C}$ by $E(\cdot \mid \mathcal{C})$ and the condition probability given $\mathcal{C}$ by $P(\cdot \mid \mathcal{C})$. The conditional entropy of a σ-field $\mathcal{A}$ given a σ-field $\mathcal{C}$ is denoted by $H(\mathcal{A} \mid \mathcal{C})$ and is defined to be

$$\sup \{E[-\textstyle\sum_{F \varepsilon \mathcal{F}} P(F \mid \mathcal{C}) \log P(F \mid \mathcal{C})]\}$$

where the supremum is taken over all finite partitions $\mathcal{F}$ of Ω into sets from $\mathcal{A}$. For properties of $H(\mathcal{A} \mid \mathcal{C})$ one may consult Jacobs [2] or Billingsley [1].

DEFINITION 1. Let I be an index set and for each i in I let $\{X_n^i : n \geqq 0\}$ be a sequence of random variables. We say that $\{X_n^i\}$ L_r-converges uniformly in i to X^i provided that for every $\epsilon > 0$ there is an $N(\epsilon)$ such that for all $n \geqq N(\epsilon)$ $\sup_i E\{|X_n^i - X^i|^r\} < \epsilon$. We say that $\{X_n^i\}$ a.s. converges uniformly in i to X^i provided that there exists a set Z of probability zero such that for every $\epsilon > 0$ and $w \not\varepsilon Z$, there exists an integer $N(\epsilon, w)$ such that $\sup_i |X_n^i(w) - X^i(w)| < \epsilon$. We shall denote these types of convergences respectively by $X_n^i \to X^i$ [L_r unif i] and $X_n^i \to X^i$ [a.s. unif i].

LEMMA 1. *Let $\{\mathcal{A}_n\}$ denote an increasing sequence of σ-fields and $\mathcal{A}$ denote the σ-field generated by $\bigcup_n \mathcal{A}_n$. If for some n, $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ then for every $\epsilon > 0$ there exists an integer N such that for all $A \varepsilon \mathcal{A}$, there exists an event $B \varepsilon \mathcal{A}_N$ such that $P(A \triangle B) < \epsilon$.*

PROOF. Since $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ for some n, $\lim_n H(\mathcal{A} \mid \mathcal{A}_n) = H(\mathcal{A} \mid \mathcal{A}) = 0$

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