The Annals of Mathematical Statistics
1969, Vol. 40, No. 3, 844–853

ON THE DISTRIBUTION OF THE SUPREMUM FUNCTIONAL FOR SEMI-MARKOV PROCESSES WITH CONTINUOUS STATE SPACES

BY LAWRENCE D. STONE

Daniel H. Wagner, Associates

1. Introduction. Let $\{X_t : t \geqq 0\}$ be a separable stochastic process and $Z(t) = \sup [X_s : 0 \leqq s \leqq t]$. In a previous paper [7] this author extended the results of Baxter [1] by characterizing the distribution of $Z(t)$ for a large class of Markov and semi-Markov processes with denumerable state spaces. These results were obtained by considering the time and place of the first jump.

The purpose of this paper is to extend the results of [7] to Markov and semi-Markov processes whose sample paths are step functions and whose state space may be an arbitrary subset of the real numbers. In order to do this we have defined semi-Markov jump processes in Section 2. These processes are a generalization of Markov jump processes as described in [4], p. 316, and semi-Markov processes as defined in [5] and [8]. Theorem 2.1 shows that these processes may also be analyzed by considering the time and place of the first jump.

In Section 3, the distribution of $Z(t)$ for a large class of semi-Markov jump processes is characterized by a recurrence relation involving operators. In the case where the process is homogeneous in space, an analog of Spitzer's identity (see [6]) is proved. In Section 4, an example is presented which shows how the recurrence relation can be used to guess the double transform of $Z(t)$ for processes which are homogeneous in space.

2. Semi-Markov jump processes. In this section we present the definition of a semi-Markov jump process and investigate briefly some of its properties.

Let $\{X_t : t \geqq 0\}$ be a separable stochastic process with random variables, X_t, defined on the probability space $(\Omega, \mathfrak{A}, P)$, and having their range in R, the real numbers. Define

$$\begin{aligned} Y_t &= t \quad \text{if} \quad X_s = X_t \qquad \text{for all} \quad 0 \leqq s \leqq t, \\ &= t - \sup [s : 0 \leqq s \leqq t; X_s \neq X_t] \qquad \text{otherwise,} \end{aligned}$$

and

$$\alpha_t = \inf [s > t; X_s \neq X_t] \qquad \text{for} \quad t \geqq 0.$$

For convenience of notation let $\alpha = \alpha_0$.

Let $R^+ = [0, \infty)$; $\mathcal{B}(R)$ and $\mathcal{B}(R^+)$ denote the σ-algebras of Borel subsets of R and R^+ respectively. Define a two-dimensional process $\{(X_t, Y_t) : t \geqq 0\}$; the

Received 2 May 1968.