The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1496–1498

DENSITY ESTIMATION BY ORTHOGONAL SERIES[1]

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There has been much discussion of density estimation by kernel methods (e.g., Whittle (1958), Parzen (1962), Watson and Leadbetter (1963)). Thus, given a random sample $x_1, \cdots, x_n$ from the density $f(x)$, the estimator

$$\hat{f}_n = 1/n \sum_{k=1}^{n} \delta_n(x - x_k) \tag{1}$$

has minimum integrated mean square error (M.I.S.E.) if

$$\phi_{\delta_n} = |\phi_f|^2/\{n^{-1}[1 + (n-1)|\phi_f|^2]\} \tag{2}$$

where $\phi_{\delta_n} = \int e^{ixt}\delta_n(x)\, dt$, $\phi_f = \int e^{ixt} f(x)dx$. Whittle introduced Bayesian arguments to find the optimum kernel; he assumed a covariance function for the values of $f(x)$ at different x values.

It is obvious that orthogonal series estimates could be used if it is assumed that $f(x) = \sum_0^\infty \alpha_m \varphi_m(x)$, where $(\varphi_m(x))$ is an orthonormal basis. Several papers (Cencov (1962), van Ryzin (1966), Schwartz (1967), and Kronmal and Tarter (1968)) have considered this possibility. The estimator that springs to mind is

$$f_n^*(x) = \sum_0^\infty \lambda_m(n)\, a_m \varphi_m(x) \tag{3}$$

where

$$a_m = n^{-1} \sum_{k=1}^{n} \varphi_m(x_k). \tag{4}$$

and the sequence $\{\lambda_m(n)\}$ is chosen to improve the properties of $f_n^*(x)$ e.g., to make $f_n^*(x)$ a M.I.S.E. estimator in its class. The papers of Cencov, van Ryzin and Schwartz use a special sequence $\{\lambda_m(n)\}$; they set $\lambda_m(n) = 1$, $m = 1, \cdots, M(n)$, $\lambda_m(n) = 0$, $m > M(n)$, and concern themselves, in part, with the determination of $M(n)$. Now

$$J = E \int (f(x) - f_n^*(x))^2\, dx = \sum_0^\infty E(\alpha_m - \lambda_m(n) a_m)^2$$
$$= \sum_0^\infty \{\alpha_m^2(1 - \lambda_m(n))^2 + n^{-1}\lambda_m^2(n) \operatorname{var}(\varphi_m(x))\}.$$

Hence

$$\lambda_m(n) = \alpha_m^2/[\alpha_m^2 + n^{-1} \operatorname{var}(\varphi_m(x))], \tag{5}$$

i.e.,

$$\lambda_m(n) = \{\alpha_m^2/E(\varphi_m^2)\}/\{n^{-1}[1 + (n-1)\, \alpha_m^2/E(\varphi_m^2)]\}. \tag{5'}$$

Received 31 May 1968.

[1] Research supported by the Office of Naval Research under contract NONR 4010(09) awarded to the Department of Statistics, The Johns Hopkins University. This paper in whole or in part may be reproduced for any purpose of the United States Government.