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DIVERGENCE PROPERTIES OF SOME MARTINGALE TRANSFORMS

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A martingale difference sequence $d = (d_n, n \geqq 1)$ relative to the sequence $(\mathfrak{F}_k, k \geqq 0)$ of σ-fields is said to satisfy condition MZ if

(i) $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ a.e. for all n
(ii) $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq K$ a.e. for some $K > 0$ and all n.

This definition is due to Richard F. Gundy, who in [3] studies transforms of martingales with difference sequences d satisfying condition MZ, that is, processes of the form $(\sum_1^n v_i d_i, n = 1, 2, \cdots)$ where $v = (v_i, i \geqq 1)$ is a sequence of functions such that v_n is $\mathfrak{F}_{n-1}$ measurable. v is called a multiplier sequence. Gundy proves that for such a transform the three sets

$$A = \{\lim \textstyle\sum_1^n v_i d_i \text{ exists and is finite}\}$$

$$B = \{\Sigma v_1^2 < \infty\}$$

$$C = \{\Sigma v_i^2 d_i^2 < \infty\}$$

are equal with probability 1. Here we prove that, as is known in some special cases, A, B, and C are equivalent to the set

$$D^+ = \{\sup \textstyle\sum_1^n v_i d_i < \infty\}$$

(and thus also to $D^- = \{\inf \sum_1^n v_i d_i > -\infty\}$). Thus the sample functions of the process $(\sum_1^n v_i d_i, n \geqq 1)$ almost surely either converge or oscillate between $-\infty$ and $+\infty$. This was conjectured by Gundy in [3]. Y. S. Chow has already given a partial answer in [2], showing the equivalence of A, B, C, and $|D| = \{\sup |\sum_1^n v_i d_i| < \infty\}$.

The definition of property MZ is extended to finite sequences of martingale differences in the obvious way. Whenever a statement like $(d_i, \mathfrak{F}_i, i \geqq 1)$ *satisfies condition* MZ is made the 0th σ-field required by the definition is to be taken as $\{\Omega, \varnothing\}$. In addition we can and do always assume without loss of generality that $E(d_1 \mid \mathfrak{F}_0) = 0$ a.e..

LEMMA 1. *Let* $\epsilon > 0$. *There is a number* $\delta_\epsilon = \delta_\epsilon(K)$ *such that if* $d = (d_1, \cdots, d_n)$ *satisfies* (i) *and* (ii) *and* $(v_1, \cdots, v_n) = v$ *is a multiplier sequence such that* $P(\sum_1^n v_i^2 > \epsilon) > \epsilon$ *then*

$$P(\sup_{1 \leqq k \leqq n} |\textstyle\sum_1^k v_i d_i| > \delta_\epsilon) > \delta_\epsilon.$$

PROOF. Suppose this is not true. For $i \geqq 1$ let $d_i = (d_{ij}, \mathfrak{F}_{ij}, 1 \leqq j \leqq n_i)$ be independent martingale difference sequences, that is let $\mathfrak{F}_{1n_1}, \mathfrak{F}_{2n_2}, \cdots$ be independent σ-fields, and let $(v_{ij}, 1 \leqq j \leqq n_i)$ be associated multiplier sequences such that for each i the conditions of the lemma are satisfied and such that

$$P(\sup_{1 \leqq k \leqq n_i} |\textstyle\sum_{j=1}^k v_{ij} d_{ij}| > 1/2^i) \leqq 1/2^i.$$