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SOME STRIKING PROPERTIES OF BINOMIAL AND BETA MOMENTS[1]

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1. Introduction. For each positive integer n, let M_n denote the convex body of n-tuples $(C_1, C_2, \cdots, C_n)$ with

$$C_i = \int_{[0,1]} x^i \, d\sigma(x), \qquad i = 1, 2, \cdots, n,$$

where σ is allowed to vary over the class of all probability measures on the Borel subsets of the unit interval [0, 1]. Detailed treatment of these spaces may be found in [2] and [3]. For the moment sequence $(C_1, C_2, \cdots)$ corresponding to an arbitrary σ, write

$$\nu_n(C_1, C_2, \cdots) = C_n$$
$$\nu_n^{\pm}(C_1, C_2, \cdots) = {}^{max}_{min} \{d: (C_1, C_2, \cdots, C_{n-1}, d) \in M_n\},$$

and take $R_n = \nu_n^+ - \nu_n^-$.

Let $M_n^0 (n > 1)$ denote that subset of M_n whose points $(C_1, \cdots, C_{n-1}, C_n)$ have $(C_1, \cdots, C_{n-1})$ interior to M_{n-1}; $M_1^0 = M_1$. In [5], we defined "normalized" moments $p_n = 1 - q_n$ on M_n^0 by taking

$$(1.1) \qquad p_n = (\nu_n - \nu_n^-)/R_n.$$

Note that $p_1, p_2, \cdots, p_{n-1}$ may be viewed as functions on M_n^0 and in appropriate context below they will be so regarded (in this connection see Corollary 1.1, page 107 of [3] and its proof). In [5] it was proved that everywhere on M_n^0

$$(1.2) \qquad R_{n+1} = \prod_{i=1}^{n} p_i q_i.$$

If we define the right hand side to be zero on $M_n - M_n^0$, (1.2) holds there as well. (1.2) and certain of its corollaries which appear in [6] (in particular Corollary 2) exhibit in an emphatic manner the fundamental nature of the normalization (1.1). The somewhat startling form of normalized binomial moments (Theorem 2), the direct connection between moments of even index and the distribution's support, and the simple form of normalized Beta moments (Theorem 3) tend strongly to reinforce this judgment.

In Section 4, two general theorems are given. The first exhibits the connection between two normalized moment sequences whose corresponding distributions on [0, 1] are symmetrically related. The second theorem exhibits the invariance of moment normalization under the standard 1–1 mapping that takes the class of all distributions on a finite interval $[a, b]$ onto the class of all distributions on [0, 1].

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