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STOCHASTIC INTEGRALS AND DERIVATIVES[1]

BY DEAN ISAACSON

Iowa State University

0. Introduction. The question we consider in this paper is whether or not the stochastic integral has a property analogous to the Fundamental Theorem of Calculus. That is, if $Y(t, \omega) = \int_0^t \phi(s, \omega)\, dM(s, \omega)$ and if $\Delta Y(t, \omega) = Y(t + \Delta t, \omega) - Y(t, \omega)$, does $\lim_{\Delta t \to 0} \Delta Y(t, \omega)/\Delta M(t, \omega) = \phi(t, \omega)$ and in what sense does the limit exist?

1. Stochastic integrals using Brownian motion as integrator. In this section we consider only stochastic integrals using Brownian motion as integrator. Throughout this paper we will let $X = (X_t, \mathfrak{F}_t, t \geqq 0)$ denote one dimensional standard Brownian motion defined on $(\Omega, \mathfrak{F}, P)$, a complete probability space. Let $\mathfrak{F}_t$ be the complete sub σ-field of $\mathfrak{F}$ generated by $\{X_s : s \leqq t\}$. (By standard Brownian motion we mean the process is normalized so that

$$\operatorname{Var}[X(t, \omega) - X(s, \omega)] = t - s \quad \text{for} \quad s < t.)$$

For notational purposes we let $\operatorname{Plim}_{\Delta t \to 0} \Delta Y(t, \omega) = H(t, \omega)$ mean that $\Delta Y(t, \omega)$ converges in probability to $H(t, \omega)$ as $\Delta t \to 0$ where we always take $\Delta t > 0$.

DEFINITION. A real valued process, $\phi(s, \omega)$, is *stochastically integrable* on R^+ with respect to $X(t, \omega)$ if:

(i) $\phi(s, \omega)$ is adapted to $\{\mathfrak{F}_s\}$.
(ii) $\phi(s, \omega)$ is measurable on $(R^+ \times \Omega, \beta(R^+) \times \mathfrak{F})$.
(iii) $\int_0^t E\,|\phi(s, \omega)|^2\, ds < \infty$ for all finite $t \geqq 0$.

Let $M_1(X)$ denote the space of all processes stochastically integrable with respect to $X(t, \omega)$. For $\phi(s, \omega) \in M_1(X)$ one can define the stochastic integral $\int_0^t \phi(s, \omega)\, dX(s, \omega)$. For a discussion of this integral see [2] or [3].

To motivate the type of answer one should expect to our question, consider the case where $\phi(s, \omega) = X(s, \omega)$. i.e., as an integrand we take Brownian motion itself. One can easily show that $X(s, \omega)$ is stochastically integrable. In fact, the integral can be evaluated.

$$\int_0^t X(s, \omega)\, dX(s, \omega) = (X^2(t, \omega)/2) - (t/2) \quad \text{([2] page 444)}.$$

Hence, if $Y(t, \omega) = (X^2(t, \omega)/2) - (t/2)$, then

$$\Delta Y(t, \omega)/\Delta X(t, \omega) = X(t, \omega) + (\Delta X(t, \omega)/2) - (\Delta t/2\Delta X(t, \omega)).$$

We now must show the last two terms on the right-hand side go to zero. Fix $t \geqq 0$. Now as $\Delta t \to 0$, one easily sees that $\Delta X(t, \omega)/2 \to 0$ a.s. and in L_2. How-

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[1] This paper is based on my Ph.D. Thesis written under the direction of Professor Steven Orey at the University of Minnesota.