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ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRIX

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1. Introduction. In our previous paper [14], we have proved the unbiasedness of the modified likelihood ratio (= modified LR) tests (i) for the equality of covariance matrix to a given matrix and (ii) for the equality of two covariance matrices. We have also shown the unbiasedness of the LR tests (iii) for sphericity and (iv) for the equality of mean vector and covariance matrix to a given vector and matrix.

In this paper asymptotic expansions of the distributions of the test criteria for (i) and (iv) both under hypothesis and alternatives are derived, by inverting the characteristic function directly. The asymptotic expansion of the non-null distribution of the LR criterion for sphericity (iii), is obtained by using the differential operator due to Welch [15], and also the limiting non-null distribution of the LR test for the equality of k covariance matrices is derived in a similar way. This method has been shown to be useful in other problems in multivariate analysis by Ito [5], Siotani [11], Okamoto [8], and others.

All the limiting non-null distributions of these test criteria are shown to be normal distributions, whereas the limiting distributions under hypothesis are χ^2–distributions as in Box [2] or Anderson ([1], Chapter 10). It may be interesting to note that the limiting non-null distribution of the likelihood ratio criterion for the multivariate linear hypothesis is noncentral χ^2, the asymptotic expansion of which was obtained by Sugiura and Fujikoshi [13].

2. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$. Let $p \times 1$ vectors $X_1, \cdots, X_N$ be a random sample from a p-variate normal distribution with unknown mean vector μ and covariance matrix Σ (positive definite). The LR criterion for testing the hypothesis $H_1: \Sigma = \Sigma_0$ against the alternatives $K_1: \Sigma \neq \Sigma_0$, for some given positive definite matrix Σ_0, is given by

$$\lambda = (e/N)^{Np/2} |S\Sigma_0^{-1}|^{N/2} \operatorname{etr} \{-(\tfrac{1}{2})\Sigma_0^{-1}S\}, \tag{2.1}$$

where etr means exp tr and $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = (1/N) \sum_{\alpha=1}^{N} X_\alpha$. This LR test is not unbiased. However, if we modify this criterion by reducing the sample size N to the degrees of freedom $n = N - 1$, it has some desirable property, that is, the unbiasedness is shown by Sugiura and Nagao [14] and the monotonicity of the power function with respect to p characteristic roots of $\Sigma\Sigma_0^{-1}$ is established by Nagao [7] and Das Gupta [4].

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