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ON THE DISTRIBUTIONS OF THE RATIOS OF THE ROOTS OF A COVARIANCE MATRIX AND WILKS' CRITERION FOR TESTS OF THREE HYPOTHESES[1]

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1. Introduction. Let $\mathbf{X}(p \times n)$ be a matrix variate with columns independently distributed as $N(\mathbf{0}, \mathbf{\Sigma})$. Then the distribution of the latent roots, $0 < w_1 \leqq \cdots \leqq w_p < \infty$, of $\mathbf{XX}'$ is first considered in this paper for deriving the distributions of the ratios of individual roots $w_i/w_j (i < j = 2, \cdots, p)$. In particular, the distributions of such ratios are derived for $p = 2$, 3 and 4. The use of these ratios in testing the hypothesis $\delta\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, $\delta > 0$ unknown, has been pointed out, where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are the covariance matrices of two p-variate normal populations.

Further, the non-central distributions of Wilks' criterion, $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1 - c_i)$, are obtained in the following cases: (1) test of $\mathbf{\Sigma}_1 = \delta\mathbf{\Sigma}_2$, $\delta > 0$ known, (2) MANOVA and (3) Canonical correlation, where c_i's stand for latent roots of a matrix arising in each of the situations. The density functions are given in terms of Meijer's G-function [12] and for $p = 2$, the density and distribution functions are explicitly evaluated. For Case (2), Pillai and Al-Ani [15] have derived the density for $p = 2$, 3 and 4 using some results on Mellin transforms [2, 3, 4], and Jouris [9] has shown by induction that the G-function can be expressed in an alternate form than given in the paper; this latter form includes as special cases the results of Pillai and Al-Ani [15].

2. Distribution of ratios of the roots of a covariance matrix. The distribution of the latent roots, $0 < w_1 \leqq w_2 \leqq \cdots \leqq w_p < \infty$ of $\mathbf{XX}'$ depends only upon the latent roots of $\mathbf{\Sigma}$ and can be given in the form (James [6])

$$(2.1)\quad K(p, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} |\mathbf{W}|^m \{\exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{W})\} \cdot \textstyle\prod_{i>j} (w_i - w_j) {}_0F_0(\tfrac{1}{2}(\mathbf{I}_p - \mathbf{\Sigma}^{-1}), \mathbf{W}), \quad 0 < w_1 \leqq w_2 \leqq \cdots \leqq w_p < \infty,$$

where

$$m = \tfrac{1}{2}(n - p - 1), \qquad K(p, n) = \Pi^{\frac{1}{2}p^2}/\{2^{\frac{1}{2}pn}\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}p)\},$$

$$\mathbf{W} = \operatorname{diag}(w_1, \cdots, w_p),$$

$$(2.2)\quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \mathbf{S}, \mathbf{T}) = \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} [(a_1)_\kappa \cdots (a_p)_\kappa]/[(b_1)_\kappa \cdots (b_q)_\kappa] \cdot C_\kappa(\mathbf{S})C_\kappa(\mathbf{T})/[C_\kappa(\mathbf{I}_p)k!]$$

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